NONLINEAR WAVES AND SPATIAL DYNAMICS

The water-wave problem is the study of the three-dimensional irrotational flow of a perfect fluid bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \eta(x, z, t)\}$ subject to the forces of gravity and surface tension. This remarkable problem, first formulated in terms of a potential function ϕ by Euler (Figure 1), has become a paradigm for most modern methods in nonlinear functional analysis and nonlinear dispersive wave theory. Its mathematical study has historically called upon many different approaches (iteration methods, bifurcation theory, complex variable methods, PDE methods, the calculus of variations, positive operator theory, topological degree theory, KAM theory, symplectic geometry, ...). In this short note I would like to illustrate the role of the water-wave problem as a paradigm in the theory of Hamiltonian systems and conservative pattern-formation problems.

$$\boxed{\begin{aligned}
&\phi_{zz} + \phi_{yy} + \phi_{zz} = 0, && 0 < y < h + \eta,\\
&\phi_y = 0, && y = 0,\\
&\phi_y = \eta_t + \eta_x \phi_x + \eta_z \phi_z, && y = h + \eta,\\
&\phi_t = -\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta &&\\
&\qquad + \sigma \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x + \sigma \left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z, && y = h + \eta
\end{aligned}}$$

Figure 1: Euler (1707–1783), who first formulated the water-wave problem (left)

Travelling water waves are solutions of the water-wave problem which are stationary in a uniformly translating reference frame, so that $\eta(x, z, t) = \eta(\xi, z)$, where $\xi = x - ct$. The resulting time-independent problem can be approached using the method of *spatial dynamics*, which was devised by K. Kirchgässner specifically with water waves in mind and has now found applications in a huge range of other problems (reaction-diffusion equations, spiral waves, mathematical biology, ...). The idea is to formulate a stationary problem as an evolutionary equation in which an unbounded *spatial* coordinate plays the role of the time-like variable. In the travelling water-wave problem one can take any horizontal direction $X = \sin\theta_2\, \xi - \cos\theta_2\, z$ as the time-like variable and formulate the equations as an evolutionary equation

$$u_X = Lu + Nu, \qquad u \in \mathcal{X}; \tag{1}$$

the infinite-dimensional phase space $\mathcal{X}$ is constructed to contain functions which are, for example, $2\pi/\nu$-periodic in a second, different horizontal direction $Z = \sin\theta_1\, \xi - \cos\theta_1\, z$. The evolutionary equation (1) is found by performing a Legendre transform upon the classical variational principle

$$\delta \int \int_0^{2\pi} \left\{ \int_0^{h+\eta} \Big(-\sin\theta_2\, \phi_X - \nu \sin\theta_1\, \phi_Z + \frac{1}{2} \left(\phi_X^2 + \phi_y^2 + \nu^2 \phi_Z^2 + 2\nu \cos(\theta_1 - \theta_2) \phi_X \phi_Z \right) \Big)\, dy \right.$$
$$\left. + \frac{1}{2} g\eta^2 + \sigma \left(\sqrt{1 + \eta_X^2 + \nu^2 \eta_Z^2 + 2\nu \cos(\theta_1 - \theta_2) \eta_X \eta_Z} - 1 \right) \right\} dZ\, dX = 0$$

for the desired wave motions. In many cases equation (1) can be treated using an invariant-manifold theory due to A. Mielke, which was again developed with this problem in mind, but is now used in a wide variety of problems (elasticity, solid mechanics, ...). This theory shows that all small, bounded solutions lie on a finite-dimensional invariant manifold and thus reduces the water-wave problem to a locally equivalent finite-dimensional Hamiltonian system; the dimension and character of this reduced system depend upon the values of the physical parameters (gravity g, surface tension σ, wave speed c, water depth h).

Two-dimensional i.e. z-independent travelling waves lend themselves naturally to an application of the spatial dynamics method with $X = \xi$. B. Buffoni, M. D. Groves & J. F. Toland showed that in a certain parameter regime the invariant manifold is four dimensional and controlled by the Hamiltonian equation

$$u'''' + Pu'' + u - u^2 = 0, \qquad P \in (-2, -2 + \epsilon).$$

Amazingly, this equation turns up in many, seemingly unrelated problems in applied science, for example in nonlinear elasticity, nonlinear optics and now nonlinear water waves. One of its most interesting features is that it exhibits *chaotic behaviour:* there is a Smale-horseshoe structure in its solution set. As a consequence, it has infinitely many *homoclinic solutions,* that is solutions which decay to zero as the time-like variable tends to infinity. The corresponding solutions of the water-wave problem are called *solitary waves* and decay to the undisturbed state of the water as $\xi \to \pm\infty$. This result shows that there are infinitely many of them; they are waves of depression with $2, 3, 4, \ldots$ large troughs separated by $2, 3, \ldots$ small oscillations, and their oscillatory tails decay exponentially to zero. Two waves from this family are sketched in Figure 2.

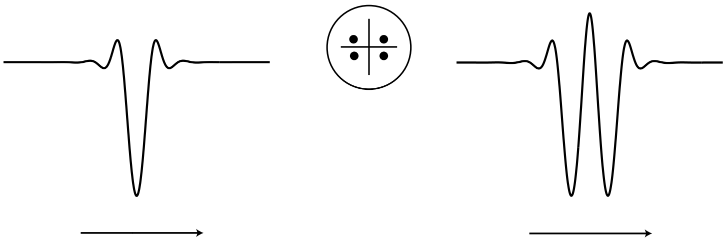

Figure 2: Two of the multi-troughed solitary waves found by B. Buffoni, M. D. Groves & J. F. Toland on a four-dimensional invariant manifold with the depicted eigenvalue structure.

The study of two-dimensional solitary waves was continued by G. Iooss & K. Kirchgässner and B. Buffoni & M. D. Groves, who noticed that there are parameter values for which the invariant manifold is four-dimensional and a *Hamiltonian-Hopf bifurcation* takes place (two nonsemisimple imaginary eigenvalues become complex as a parameter is varied). Hamiltonian-Hopf bifurcations are well-known to researchers in the field of celestial mechanics, where they occur in the restricted three-body problem for the planar motion of a light body orbiting the centre of mess of two heavy bodies; the Hamiltonian-Hopf bifurcation occurs for a certain value of the mass ratio of the two heavy particles (Routh's ratio). Iooss & Kirchgässner used the Birkhoff normal form to show that Hamiltonian-Hopf bifurcations generate homoclinic solutions which take the form of periodic wave trains modulated by exponentially decaying envelopes (Figure 3). Buffoni & Groves showed that there are in fact infinitely many such solutions which resemble multiple copies of Iooss & Kirchgässner's solutions; their proof is based upon modern methods from the calculus of variations (mountain-pass arguments and the concentration-compactness principle) and the topological degree. These results are not restricted to the water-wave problem in which they emerge; they provide dramatic new solutions to the three-body problem and indeed Hamiltonian-Hopf bifurcations have been detected in a range of situations (Taylor-Couette flows, nonlinear elasticity,. . .).

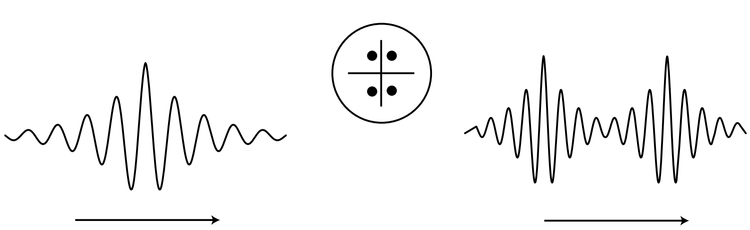

Figure 3: Two of the multi-packet solitary waves found by B. Buffoni & M. D. Groves on a four-dimensional invariant manifold with the depicted eigenvalue structure.

M. D. Groves & A. Mielke were the first reseachers to apply the spatial dynamics method to three-dimensional travelling waves. They took $X = \xi$ and $Z = z$, so that the waves are periodic in the direction transverse to the direction in which they propagate. In particular, they identified a parameter region in which the invariant manifold is six-dimensional and is controlled by a Hamiltonian-Hopf bifurcation combined with a pair of simple imaginary eigenvalues. Iooss & Kirchgässner had shown how a Hamiltonian-Hopf bifurcation leads to solitary waves, and the classical Lyapunov centre theorem states that a pair of simple imaginary eigenvalues is associated with periodic waves. Groves & Mielke showed how to combine these results to obtain a *generalised solitary wave* which decays to a periodic wave (in fact a z-independent periodic wave) as $\xi \rightarrow \pm\infty$ (see Figure 4); their result can be extended to multi-packet generalised solitary waves using the method of B. Buffoni & M. D. Groves and applies to any six-dimensional Hamiltonian system for which this bifurcation occurs.

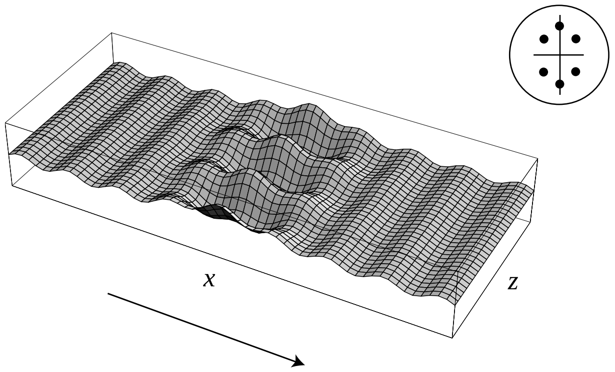

Figure 4: A generalised solitary wave found by M. D. Groves & A. Mielke on a six-dimensional invariant manifold with the depicted eigenvalue structure.

M. D. Groves & M. Haragus have recently classified all the possible bifurcation scenarios for three-dimensional travelling waves using the spatial dynamics method. In particular, they compiled a catalogue of three-dimensional waves which have solitary-wave or generalised solitary-wave profiles in a distinguished horizontal direction (the time-like direction); some of the waves are rather exotic, as Figure 5 shows.

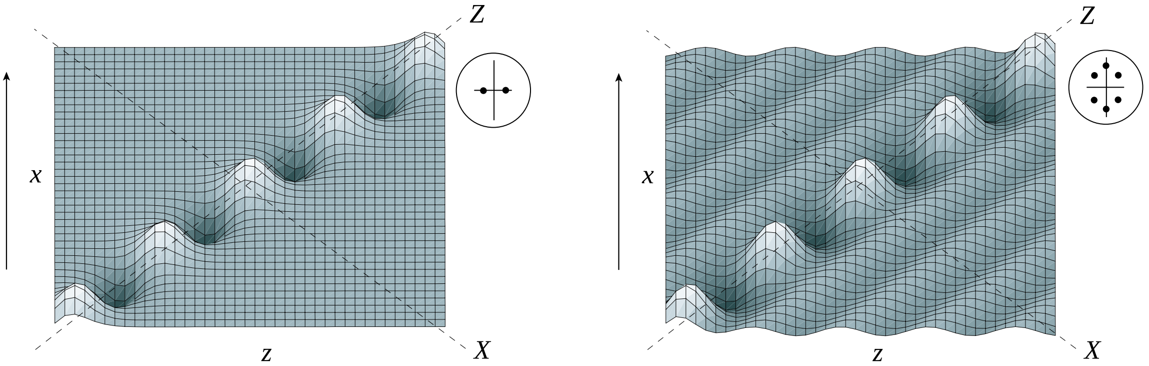

Figure 5: Two examples from the catalogue of three-dimensional travelling waves compiled by M. D. Groves & M. Haragus. The wave on the left has a solitary-wave profile in X while the wave on the right has a generalised solitary-wave profile in X; both waves are periodic in Z and move in the x-direction with constant speed.

Groves & Haragus also examined doubly periodic travelling waves using spatial dynamics. Periodicity in the Z-direction is built into the method, so that doubly periodic waves are found as solutions of the reduced Hamiltonian system which are periodic in the time-like direction X. Such solutions are found using the classical Lyapunov centre theorem, and depending upon the physical parameters one encounters all possible cases: nonresonant eigenvalues, semisimple eigenvalue resonances, nonsemisimple eigenvalue resonances and equal or opposite Krein signatures! Doubly periodic surface waves (Figure 6) and periodic motion of heavenly bodies (the n-body problem in celestial mechanics) are, according to the above observations, two aspects of the same mathematical theory, namely finite-dimensional Hamiltonian systems and the Lyapunov centre theorem.

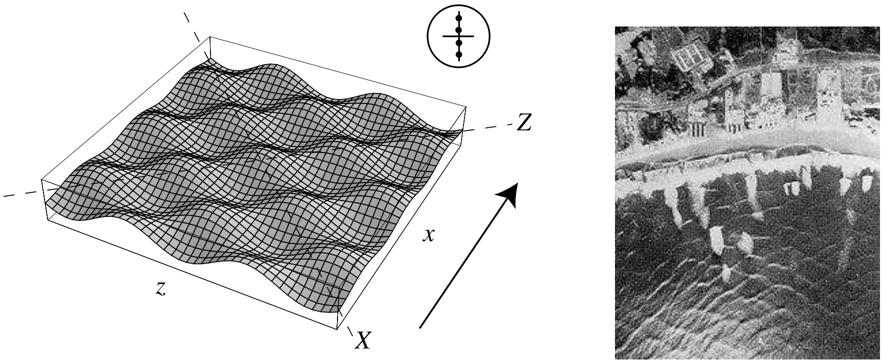

Figure 6: The doubly periodic wave on the left is constructed using the Lyapunov centre theorem on a four-dimensional invariant manifold. Hexagonal doubly periodic waves are often seen in nature, as this ariel photograph on the right shows; they can be explained mathematically by this procedure.

I would like to conclude with two examples in which a reduction to a finite-dimensional invariant manifold is not possible, but the spatial dynamics method nevertheless leads to an existence theroem for a travelling water wave. The first example is a *dimension-breaking phenomenon,* a term which describes the spontaneous emergence of a spatially inhomogeneous solution of a partial differential equation from a solution which is homogeneous in one or more spatial dimensions. A relevant example would be the solitary-wave solution of the KdV equation, which is one of a family of solutions of the KP equation which have a solitary-wave profile in ξ and are periodic in z; these *periodically modulated solitary waves* emerge from the KdV solitary wave (a *line solitary wave*) in a dimenson-breaking bifurcation. Turning to travelling water waves, suppose we take $X = z$ in the spatial dynamics method and choose the phase space $\mathcal{X}$ to accomodate waves which are evanescent as $\xi \to \pm\infty$. Any line solitary wave is therefore an equilibrium solution of (1) in this framework. The spectrum of the linear operator L at this equilibrium typically consists of a pair of purely imaginary eigenvalues and continuous spectrum covering the whole of the real axis. Without the continuous spectrum, the Lyapunov centre theorem would give a family of periodic solutions which would correspond to periodically modulated solitary waves. M. D. Groves, M. Haragus & S.-M. Sun, showed how to modify the Lyapunov centre theorem to overcome the difficulty due to the continuous spectrum and obtain the same result (see Figure 7).

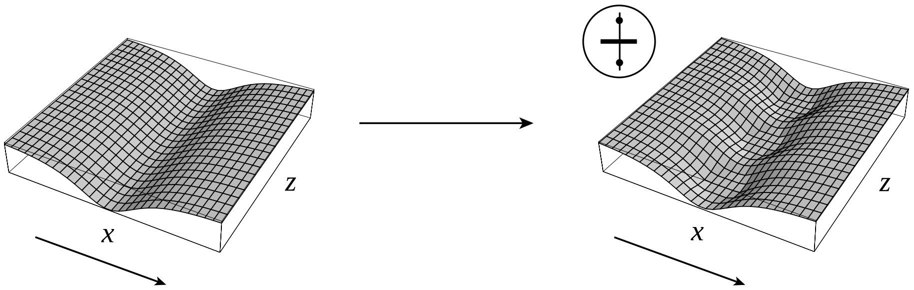

Figure 7: The periodically modulated solitary wave on the right emerges from the line solitary wave on the left in a dimension-breaking bifurcation.

The second example concerns wave motions of the form $\eta(x,z,t) = \eta(x - c_{\mathrm{g}}t, x - c_{\mathrm{p}}t)$ which are periodic in their second argument and have a solitary or generalised solitary-wave profile in their first (an informative example is $\operatorname{sech}(x - c_{\mathrm{g}}t)\sin k(x - c_{\mathrm{p}}t)$). These waves are not travelling waves, rather they consist of an envelope which advances with the velocity c_{g} (the ‘group velocity’) and modulates a periodic wavetrain moving with velocity c_{p} (the ‘phase velocity’). An example of such a *modulating pulse* is shown in Figure 8. M. D. Groves & G. Schneider applied the spatial dynamics method with $X = x - c_{\mathrm{g}}t$ and $Z = x - c_{\mathrm{p}}t$ and showed that there is an infinite-dimensional invariant manifold, the linearised flow on which is controlled by two real and infinitely many purely imaginary eigenvalues. In a Birkhoff normal form approximation the real and imaginary parts decouple, the former associated with a pulse and the latter with small-amplitude almost periodic motions. Groves & Schneider showed how to use energy estimates to glue these structures together in the full problem and create modulating pulses which decay to small, non-zero disturbances as $X \to \pm\infty$. The same result holds for other equations with this eigenvalue structure, notably nonlinear wave equations such as the Sine-Gordon and ϕ^4 equations.

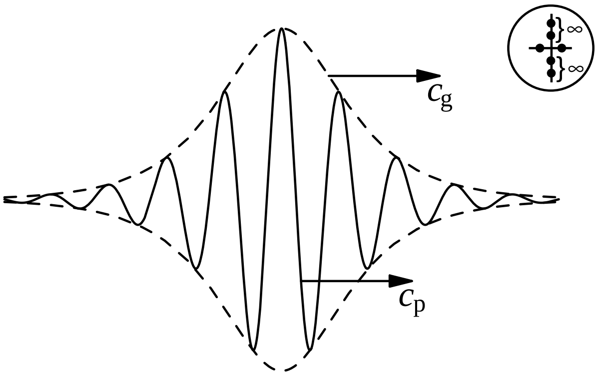

Figure 8: A modulating pulse which decays to a small-amplitude disturbance at large distances

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