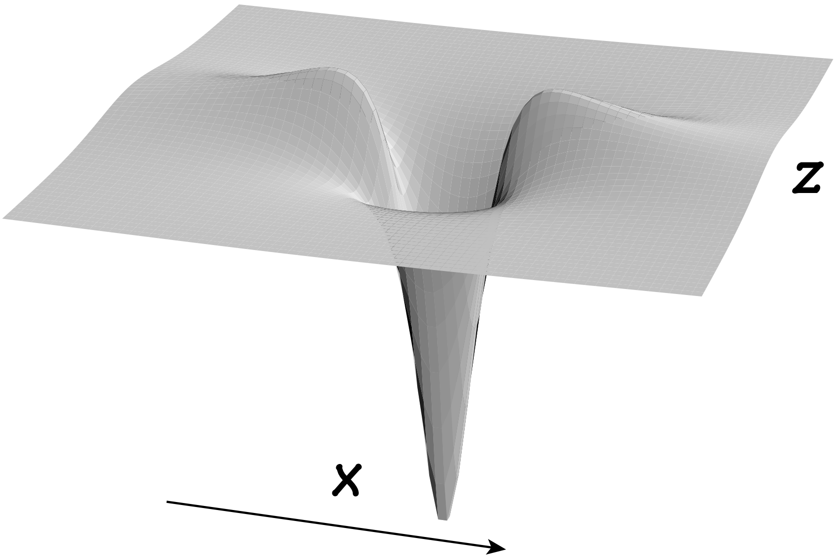
THE WATER-WAVE PROBLEM

- Domain $\{-1 < y < \eta(x,z,t)\}$, velocity potential ϕ.
- Find a solution of the water-wave equations with

$$\eta = \eta(x-ct,z) \to 0, \qquad |(x-ct,z)| \to \infty.$$



Minimise the *energy*

$$H(\eta,\xi) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta \right\}$$

subject to fixed *momentum*

$$I(\eta,\xi) = \int_{\mathbb{R}^2} \eta_x \xi = 2\mu, \qquad 0 < \mu \ll 1,$$

where $\xi = \phi|_{y=\eta}$ and

$$G(\eta)\xi = \sqrt{1+\eta_x^2+\eta_z^2}\,\phi_n|_{y=\eta}, \qquad \begin{aligned} \Delta\phi &= 0, \\ \phi|_{y=\eta} &= \xi, \\ \phi_y|_{y=-1} &= 0 \end{aligned}$$

- H and I are conserved quantities
- Yields *conditional, energetic, orbital* stability of the *set* of minimisers.

MOTIVATION

The KP-I equation

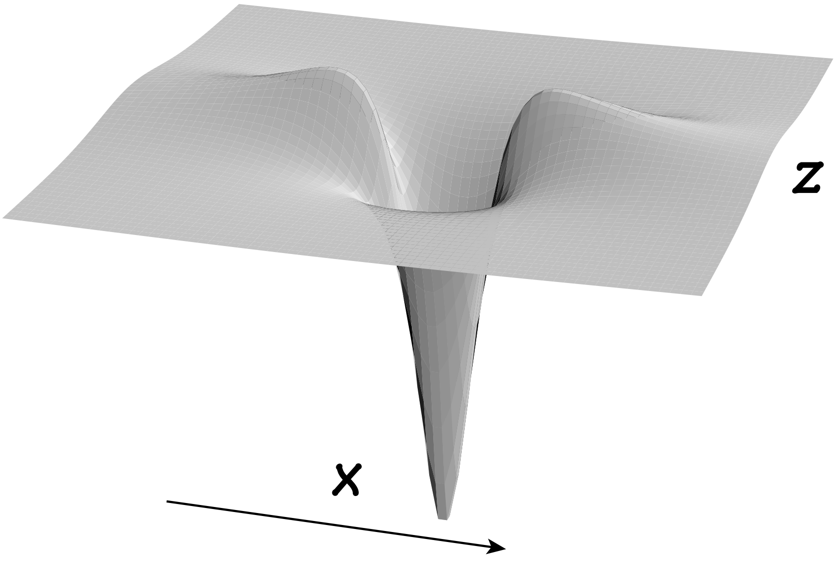
$$\partial_{xx}\left(u_{xx} - u - \frac{3}{2}u^2\right) - u_{zz} = 0 \qquad (\star)$$

is a model equation for long waves with $\beta > 1/3$ and

$$\eta(x,z) = \mu^2 u\left(\frac{\mu x}{2(\beta - 1/3)^{1/2}}, \mu^2 z\right) + O(\mu^3).$$

- $(\star)$ has the explicit fully-localised solitary-wave solution

$$u(x,z) = -8\frac{3 - x^2 + z^2}{(3 + x^2 + z^2)^2}$$



REFORMULATION

Minimise

$$H(\eta,\xi) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\xi G(\eta)\xi + \frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta \right\}$$

subject to fixed

$$I(\eta,\xi) = \int_{\mathbb{R}^2} \eta_x \xi = 2\mu, \qquad 0 < \mu \ll 1$$

- Fix η and minimise $H(\eta,\xi)$ over $I(\eta,\xi) = 2\mu$. There is a unique minimiser ξ_η with

$$G(\eta)\xi_\eta = \lambda_\eta \eta_x$$

- Minimise

$$\begin{aligned} J(\eta) &= H(\eta,\xi_\eta) \\ &= \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)}, \end{aligned}$$

where

$$\mathcal{K}(\eta) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\eta^2 + \beta\sqrt{1+\eta_x^2+\eta_z^2} - \beta \right\},$$

$$\mathcal{L}(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} \eta_x G(\eta)^{-1}\eta_x$$

We show that

- $J(\eta)$ has a minimiser
- Any minimising sequence converges (up to subsequences and translations)

MATHEMATICAL FRAMEWORK

Minimise

$$\mathcal{J}(\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)},$$

where

$$\mathcal{K}(\eta) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}\eta^2 + \beta\sqrt{1 + \eta_x^2 + \eta_z^2} - \beta \right\},$$

$$\mathcal{L}(\eta) = \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} \eta K(\eta)\eta}_{\substack{>0 \\ \text{for } \eta \neq 0}}$$

and

$$K(\eta)\xi = -\partial_x(G(\eta)^{-1}\xi_x)$$

- Function spaces: $H^t(\mathbb{R}^2)$ is the space of $u(x, z)$ with finite norm

$$\|u\|_t^2 := \int_{\mathbb{R}^2} (1 + |\mathbf{k}|^2)^t \, |\hat{u}|^2.$$

- <u>Theorem</u>

  Fix $s > 1$. The linear operator

$$K(\eta) : H^{s+1}(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$$

  is an analytic function of $\eta \in H^{s+3/2}(\mathbb{R}^2)$.

We seek a minimiser of $\mathcal{J}(\eta)$ in $H^3(\mathbb{R})^2$.

MINIMISATION PROCEDURE

Minimise

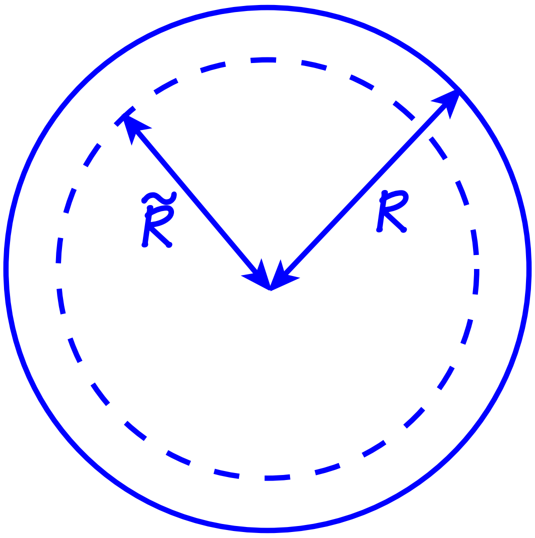
$$J(\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)}$$

- J is analytic on $\{0 < \|\eta\|_{s+3/2} < R\}$ for $s > 1$
- Quasilinear structure

We regularise and penalise the functional:

$$J_\varepsilon(\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)} + \underbrace{\varepsilon\|\eta\|_3^2}_{\text{regularise}} + \underbrace{\rho(\|\eta\|_3^2)}_{\text{penalise}}, \quad 1 < s < \tfrac{3}{2}$$

- J_ε has a semilinear structure
- Minimisers do not occur on the boundary:
  - ρ is smooth and increasing
  - $\rho(t) = 0$ for $0 \leq t < \tilde{R}$
  - $\rho(t) \to \infty$ as $t \uparrow R$



MINIMISATION PROCEDURE

- Pretend that $\mathbb{R}^2$ is bounded!

$$J_\varepsilon(\eta) = \underbrace{\mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)},}_{\substack{\text{Defined}\\ \text{on } H^{s+3/2}}} + \underbrace{\varepsilon\|\eta\|_3^2 + \rho(\|\eta\|_3^2),}_{\substack{\text{Defined}\\ \text{on } H^3}} \qquad 0 < \|\eta\|_3 < R$$

- J_ε has a minimiser $\eta_\varepsilon \neq 0$:
  - There exists a minimising sequence $\{\eta_n\}$, $J(\eta_n) \longrightarrow \inf J(\eta)$
  - A weak lower-semicontinuity argument shows that η_n converges weakly to η_ε

- η_ε lies in the region unaffected by the penalisation:
  - A priori estimates show that

$$J_\varepsilon'(\eta) = 0,\ J(\eta) < 2\mu \quad \Rightarrow \quad \|\eta\|_3^2 \leq c\mu$$

  - Motivated by the KP scaling, we find that

$$J(\eta^\star) < 2\mu, \qquad \eta^\star(x,z) = \mu^2\phi(\mu x, \mu^2 z),$$

  where ϕ is a test function

- As $\varepsilon \downarrow 0$ via a sequence $\{\varepsilon_n\}$, $\eta_{\varepsilon_n} \rightharpoonup \eta$, where η is a minimiser of J with $\|\eta\|_3 \leq c\mu$.

THE UNBOUNDED DOMAIN

$$J(\eta) = \mathcal{K}(\eta) + \frac{\mu^2}{\mathcal{L}(\eta)}, \qquad \|\eta\|_3 < R$$

We show that

- J has a minimiser η with $\|\eta\|_3^2 \leq c\mu$.
- Any minimising sequence $\{\eta_n\}$ for J with
  $$\|\eta_n\| \leq \tilde{R} < R$$
  converges to a minimiser.

Use the modified functional

$$J_P(\eta) = \mathcal{K}(\chi_P \eta) + \frac{\mu^2}{\mathcal{L}(\chi_P \eta)},$$

where χ_P is a smooth cut-off function with support in $\{(x, z) \in (-P/2, P/2)^2\}$.

- Our previous method yields a minimiser η_P for J_P with $\|\eta_P\|_3^2 \leq c\mu$.
- Let $P \longrightarrow \infty$ via a sequence $\{P_n\}$, so that $\{\eta_{P_n}\}$ is a minimising sequence for $J(\eta)$.

Use *concentration-compactness*. Difficulties:

- Nonlocal operators
- A nonhomegeneous nonlinearity