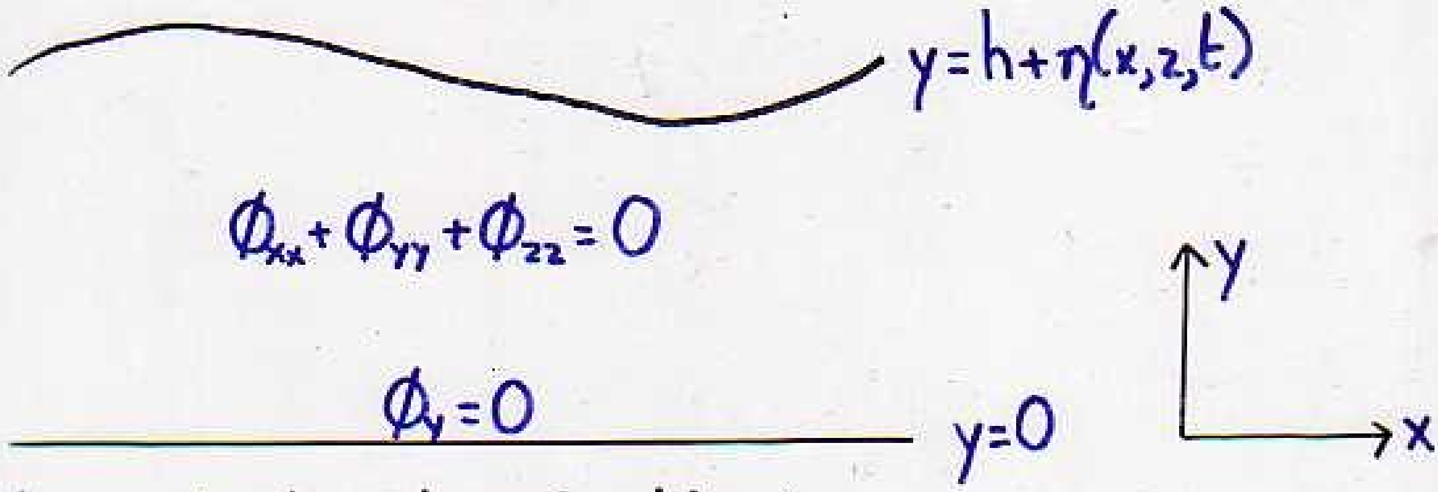
THE WATER-WAVE PROBLEM



kinematic boundary condition:

$$\eta_t = \phi_y - \eta_x \phi_x - \eta_z \phi_z$$

dynamic boundary condition:

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \sigma\left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0$$

Difficulties:

- A free-boundary problem
- Nonlinear boundary conditions

Travelling waves:

$$\eta(x,z,t) = \eta(x-ct,z), \quad \phi(x,y,z,t) = \phi(x-ct,y,z)$$

Solitary waves:

$$\eta(x-ct,z) \to 0 \quad \text{as } x-ct \to \pm\infty$$

Parameters: $\alpha = gh/c^2$, $\beta = \sigma/hc^2$

KP EQUATION

Valid for $\alpha \sim 1$, $\beta > 1/3$:

$$\partial_{xx}\left(u_{xx} - u - \frac{3}{2}u^2\right) - u_{zz} = 0$$

A family of explicit solitary-wave solutions:

$$u_\varepsilon(x,z) = -\frac{4(1-\varepsilon^2)}{4-\varepsilon^2} \cdot \frac{1-\varepsilon \cosh a_\varepsilon x \cos \omega_\varepsilon z}{(\cosh a_\varepsilon x - \varepsilon \cos \omega_\varepsilon z)^2}, \quad \varepsilon \in (0,1)$$

$$a_\varepsilon = \left(\frac{1-\varepsilon^2}{4-\varepsilon^2}\right)^{1/2}, \quad \omega_\varepsilon = \frac{(3(1-\varepsilon^2))^{1/2}}{4-\varepsilon^2}$$

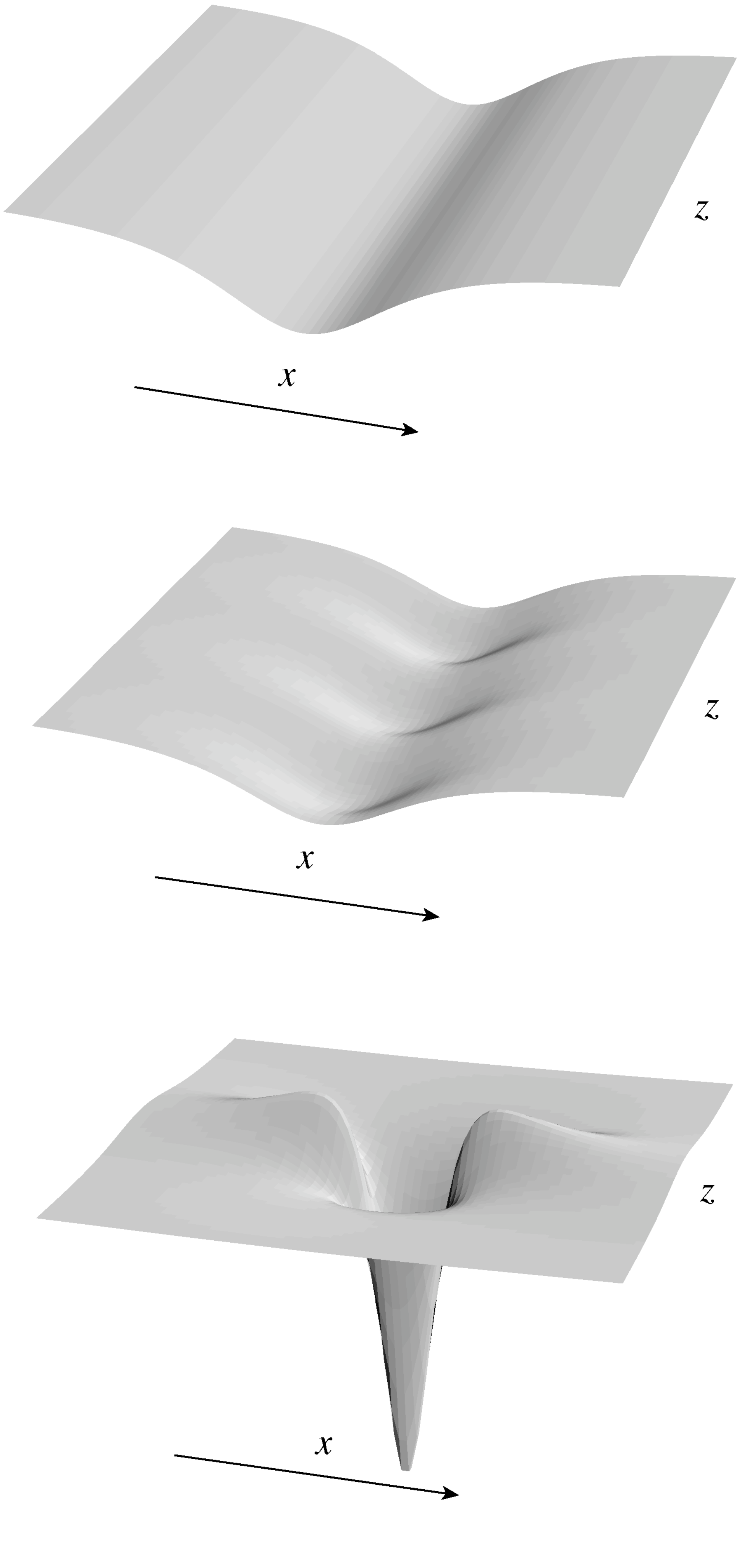
- These are periodically modulated solitary waves:

- A line solitary wave:

$$u_0(x,z) = -\operatorname{sech}^2\left(\frac{x}{2}\right)$$

- A fully localised solitary wave:

$$u_1(x,z) = -\frac{1-x^2+3z^2}{(1+x^2+3z^2)^2}$$


z
x
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A VARIATIONAL PRINCIPLE

Luke's variational principle:

$$\delta \int_{\mathbb{R}^2} \Big\{ \int_0^{1+\eta(x,z)} \left(-\phi_x + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2)\right) dy + \tfrac{1}{2}\alpha\eta^2 + \beta\left[\sqrt{1+\eta_x^2+\eta_z^2} - 1\right] \Big\}\, d(x,z) = 0$$

New variables

$$\tilde{y} = y/(1+\eta), \qquad \Phi(x,\tilde{y},z) = \phi(x,y,z)$$

$$\Rightarrow \delta \mathcal{J} = 0, \qquad \mathcal{J} = \int_{\mathbb{R}^2} F(\eta,\Phi)\, d(x,z)$$

- Evolutionary equations:

$$\delta \int_{-\infty}^{\infty} F(\eta,\Phi,\eta_x,\Phi_x)\,dx = 0 \qquad \delta \int_{-\infty}^{\infty} F(\eta,\Phi,\eta_z,\Phi_z)\,dz = 0$$

Legendre transform

$$u_x = Lu + N(u) \qquad\qquad u_z = Lu + N(u)$$

$$u = (\eta, \Phi, \partial_{\eta_x} F, \partial_{\Phi_x} F) \qquad u = (\eta, \Phi, \partial_{\eta_z} F, \partial_{\Phi_z} F)$$

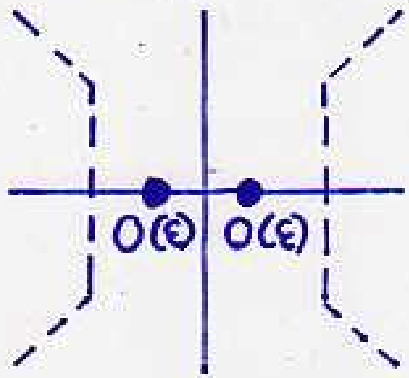
- Apply the direct methods of the calculus of variations to $\mathcal{J}$

LINE SOLITARY WAVES

Spatial dynamics, x as 'time':

$$u_x = Lu + N(u) \qquad (*)$$

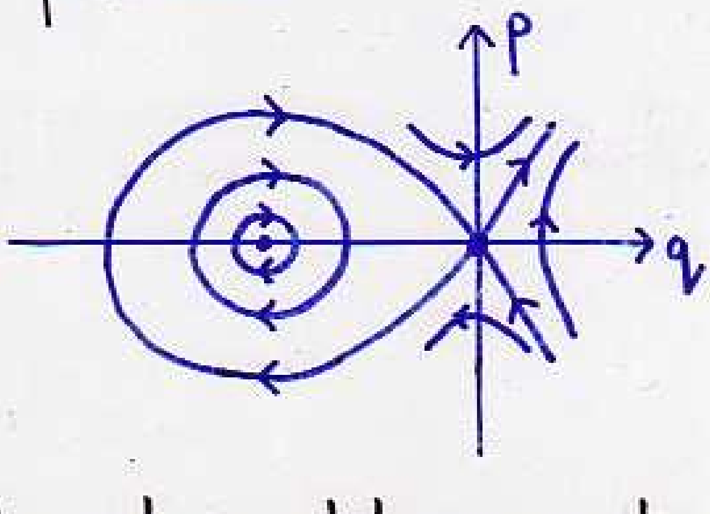
Spectrum of L for $\beta > 1/3$, $\alpha = 1+\varepsilon$:



Kirchgässner reduction: (*) has a two-dimensional invariant manifold with flow

$$q_x = p + O(\varepsilon)$$
$$p_x = q + \tfrac{3}{2}q^2 + O(\varepsilon)$$

Phase portrait:



The homoclinic solution is a line solitary wave

PERIODICALLY MODULATED SOLITARY WAVES

Spatial dynamics, z as 'time':

$$u_z = Lu + N(u) \qquad u \in X \qquad (1)$$

X is a space of functions which decay as $x \to \pm\infty$

- Equilibrium solutions of (1) $\leftrightarrow$ Line solitary waves

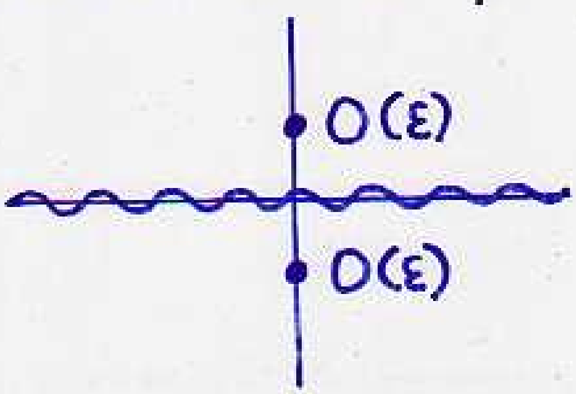
Write

$$u(x,z) = u^*_{LSW}(x) + w(x,z)$$

and study

$$w_z = L^* w + N^*(w) \qquad (2)$$

Spectrum of L^* for $\beta > 1/3$, $\alpha = 1+\varepsilon$:



Lyapunov centre theorem (Devaney, Iooss):

- $\pm in\omega \notin \sigma(L^*)$, $n \in \mathbb{N}$
- $L^* u = -N^*(w)$ solvable for all w

The periodic solutions of (2) are periodically modulated solitary waves

VARIATIONAL PRINCIPLE

$$\delta \int_{\mathbb{R}^2} \left\{ \int_0^1 \left(\frac{1}{2}\left[\Phi_x - \frac{y\Phi_y\eta_x}{1+\eta}\right]^2 + \frac{1}{2}\left[\Phi_z - \frac{y\Phi_y\eta_z}{1+\eta}\right]^2 + \frac{\Phi_y^2}{2(1+\eta)^2}\right)(1+\eta)\,dy \right.$$
$$\left. + \eta_x \Phi\big|_{y=1} + \frac{1}{2}(1+\varepsilon)\eta^2 + \beta\left[\sqrt{1+\eta_x^2+\eta_z^2} - 1\right]\right\} dx\,dz = 0$$

- Quasilinear structure

Euler-Lagrange equation for η:

$$(1+\varepsilon+\partial_x^2+\partial_z^2)\eta = \Phi_x|_{y=1} + \mathcal{F}_1(\eta,\Phi)$$

- We can solve $\eta=\eta(\Phi)$
- Variational reduction possible

Euler-Lagrange equation for Φ:

$$\begin{aligned} -\Phi_{xx} - \Phi_{yy} - \Phi_{zz} &= \mathcal{F}_2(\eta,\Phi), && 0<y<1, \\ \Phi_y &= 0, && y=0, \\ \Phi_y + \eta_x &= \mathcal{F}_3(\eta,\Phi), && y=1 \end{aligned}$$

- We use a Green's function to convert this into an integral equation for Φ:

$$\hat{\Phi} = -\int_0^1 G(\xi,y)\hat{\mathcal{F}}_2(\eta(\Phi),\Phi)\,d\xi - G(1,y)\hat{\mathcal{F}}_4(\eta(\Phi),\Phi)$$

We use a modification of the variational Lyapunov-Schmidt reduction.

REDUCTION

$$\hat{\Phi} = -\int_0^1 G(\xi, y)\hat{\mathcal{F}}_2(\Phi)\,d\xi - G(1,y)\hat{\mathcal{F}}_4(\Phi)$$

Decomposition:

$$G = -\alpha[k_z^2(1+\varepsilon) + k_x^2 + c_0(k_x^2+k_z^2)^2 + c_1(k_x^2+k_z^2)^3]^{-1} + G_1$$

$$\Phi(x,y,z) = \Phi_1(x,z) + \Phi_2(x,y,z)$$

$$\hat{\Phi}_1 = \alpha[\dots]^{-1}\left[\int_0^1 \hat{\mathcal{F}}_2(\Phi_1+\Phi_2)\,d\xi + \hat{\mathcal{F}}_4(\Phi_1+\Phi_2)\right]$$

$$\hat{\Phi}_2 = -\int_0^1 G_1(\xi,y)\hat{\mathcal{F}}_2(\Phi_1+\Phi_2)\,d\xi - G_1(1,y)\hat{\mathcal{F}}_4(\Phi_1+\Phi_2)$$

- We can solve $\Phi_2 = \Phi_2(\Phi_1)$ (two derivatives)

Reduced equation for Φ_1:

$$\frac{1}{\alpha}[\partial_z^2(1+\varepsilon) + \partial_x^2 - c_0(\partial_x^2+\partial_z^2)^2 + c_1(\partial_x^2+\partial_z^2)^3]\Phi_1$$
$$= \int_0^1 \mathcal{F}_2(\Phi_1 + \Phi_2(\Phi_1))\,d\xi + \mathcal{F}_4(\Phi_1 + \Phi_2(\Phi_1))$$

Variational structure:

$$\delta[\overset{3}{Q_1}(\Phi_1) + \overset{2}{Q_2}(\Phi_2(\Phi_1)) + \overset{2}{V_3}(\Phi_1 + \Phi_2(\Phi_1))] = 0$$

Q-form for Φ_1 Q-form for Φ_2 'Nonlinear' part of V

$$D(\Phi_1)\delta\Phi_1 + BVP(\Phi_2)\Phi_2'(\delta\Phi_1)$$
$$= \int_0^1 \mathcal{F}_2.(\delta\Phi_1 + \Phi_2'(\delta\Phi_1))\,d\xi + \mathcal{F}_4.(\delta\Phi_1 + \Phi_2'(\delta\Phi_1))$$

- Semilinear structure