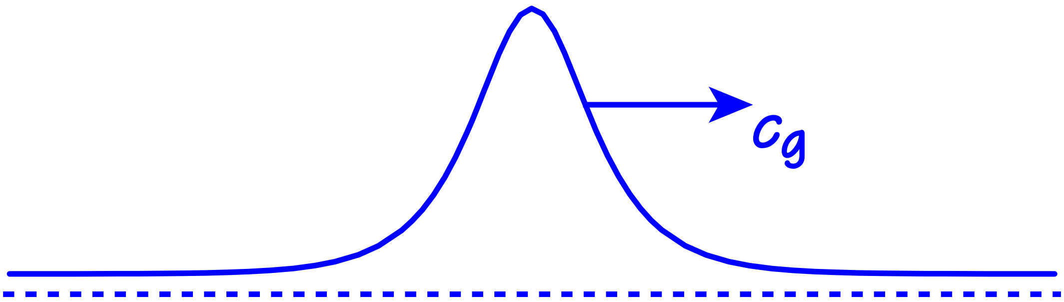
MODULATING PULSES

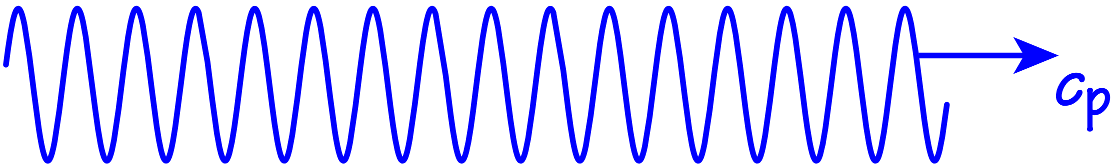
A pulse:

$$\operatorname{sech}(x - c_g t)$$



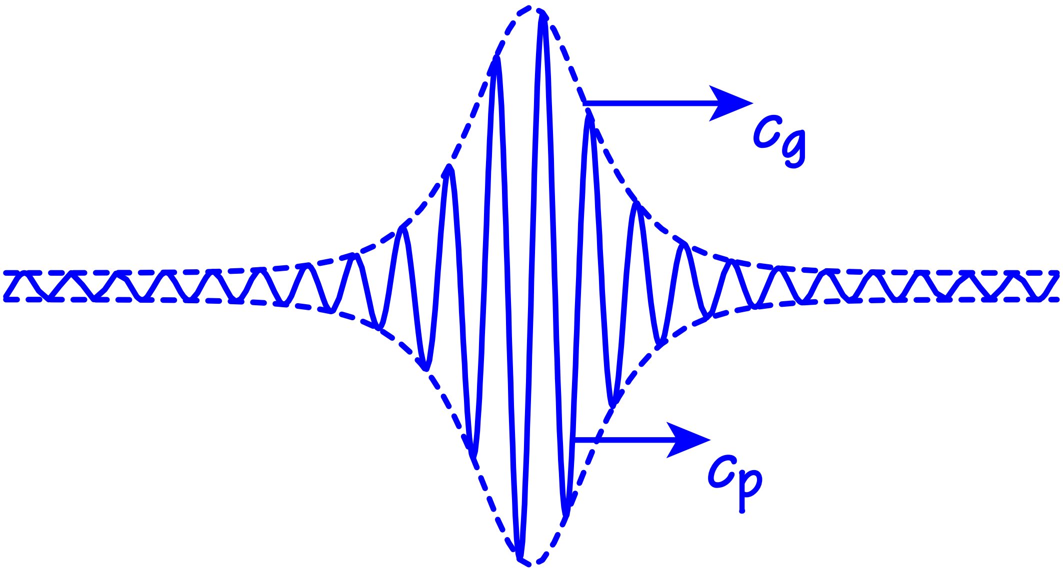
A wavetrain:

$$\sin k_0(x - c_p t)$$



A modulating pulse:

$$\operatorname{sech}(x - c_g t)\sin k_0(x - c_p t)$$



WAVE EQUATIONS

We study the equation

$$\partial_t^2 u = \partial_x^2 u - u + f_1(u, \partial_x u, \partial_t u)\partial_x^2 u + f_2(u, \partial_x u, \partial_t u),$$

where

- f_1, f_2 are analytic
- $f_i(a, -b, -c) = f_i(a, b, c), \qquad i = 1, 2$

Linear dispersion relation for $\sin k(x - c_p' t)$:

$$c_p' = (1 + k^2)^{1/2}/k$$

$$c_g' = \frac{d}{dk}(kc_p') = \frac{1}{c_p'}$$

We look for modulating pulse solutions of the form

$$\begin{aligned} u(x,t) &= v_1(x - c_g t, k_0(x - c_p t)), \\ &\quad \text{“}\operatorname{sech}(x - c_g t) \sin k_0(x - c_p t)\text{”} \end{aligned}$$

where

- $v_1(\xi, \eta)$ is 2π-periodic in η
- $c_p = c_p' + \gamma_1 \varepsilon^2, \qquad 0 < \varepsilon \ll 1$
- $c_g = 1/c_p$

SPATIAL DYNAMICS

We formulate the equation as a system for

$$v = (v_1, v_2), \qquad v_2 = \partial_\xi v_1,$$

so that

$$\begin{aligned} \partial_\xi v_1 &= v_2, \\ \partial_\xi v_2 &= -c_3^\varepsilon k_0^2 \partial_\eta^2 v_1 - c_4^\varepsilon v_1 \\ &\qquad + g_0^\varepsilon(v)\partial_\eta^2 v_1 + g_1^\varepsilon(v) + g_2^\varepsilon(v)\partial_\eta v_2, \end{aligned}$$

where

$$g_j^\varepsilon(v) = g_j^\varepsilon(v_1, \partial_\eta v_1, v_2), \qquad j = 1, 2, 3.$$

Phase space:

- $X^s = H^{s+1}_{\mathrm{per}}(0, 2\pi) \times H^s_{\mathrm{per}}(0, 2\pi), \qquad s > 0$
- The vector field has domain X^{s+1}
- $g_0^\varepsilon, g_1^\varepsilon, g_2^\varepsilon$ are bounded $X^s \longrightarrow H^s_{\mathrm{per}}(0, 2\pi)$

Reversibility:

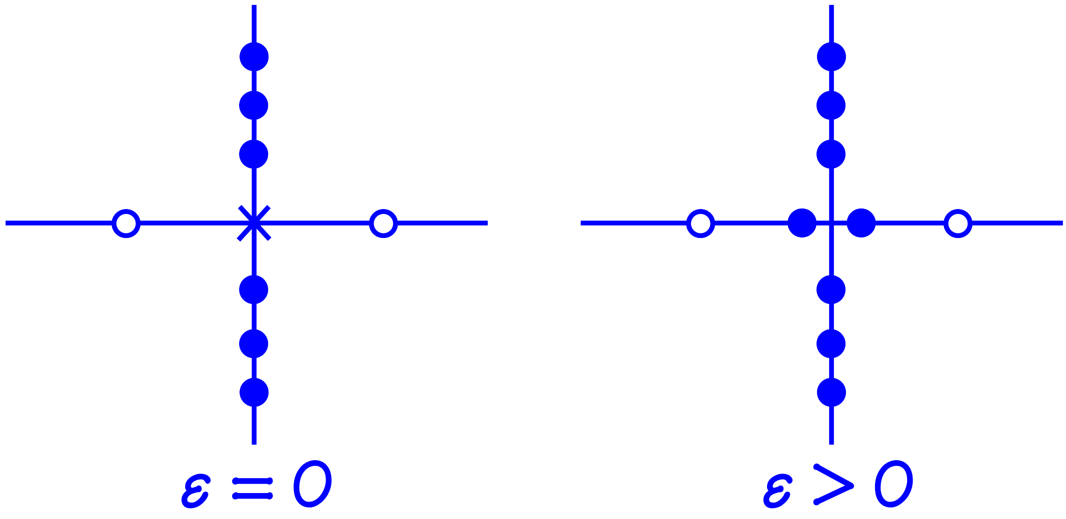
- The system is invariant under

$$\xi \mapsto -\xi, \qquad (v_1, v_2) \mapsto S(v_1, v_2),$$

$$S(v_1(\eta), v_2(\eta)) = (v_1(-\eta), -v_2(-\eta))$$

- Symmetric solutions are invariant under this transformation

SPECTRAL ANALYSIS



- Infinitely many imaginary eigenvalues $\pm i\omega_m$, $\omega_m \sim m$

Notation:

$$z = P_{wh}(v), \qquad q = P_{sh,c}(v)$$

Write as a coupled system

$$\partial_\xi z = Kz + F^\varepsilon(z,q), \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \partial_\xi q_1 &= q_2, \\ \partial_\xi q_2 &= -c_3^\varepsilon k_0^2 \partial_\eta^2 q_1 - c_4^\varepsilon q_1 + \overbrace{P_{sh,c}(g_1^\varepsilon(z,q))}^{= g_3^\varepsilon(z,q) + h^\varepsilon(z)} \\ &\quad + P_{sh,c}(g_0^\varepsilon(z,q)\partial_\eta^2 q_1) + P_{sh,c}(g_2^\varepsilon(z,q)\partial_\eta q_2), \end{aligned}$$

where $g_3^\varepsilon(z,0) = 0$.

A SIMPLIFIED PROBLEM

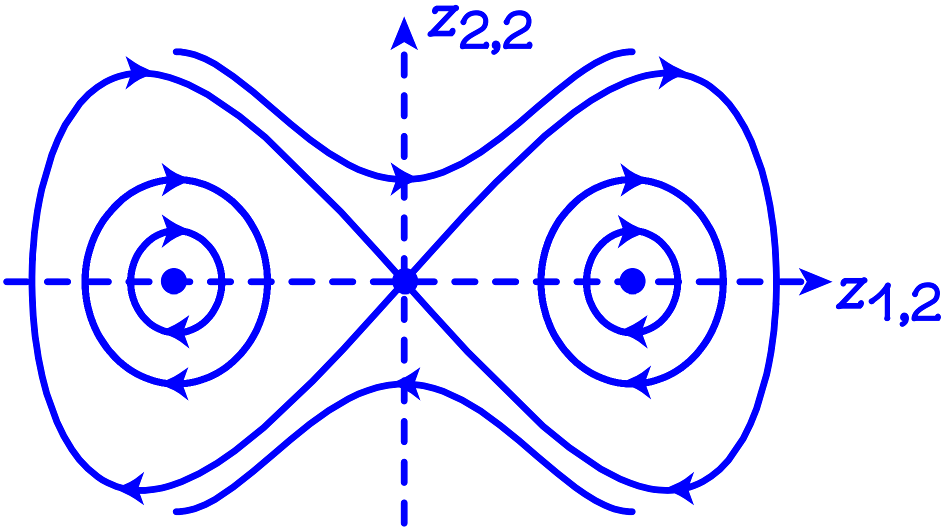
- $h^\varepsilon(z) = 0 \;\Rightarrow\; \{q = 0\}$ is invariant
- The flow in $\{q{=}0\}$ is controlled by the fourth-order equation

$$\partial_\xi z = Kz + \underbrace{F^\varepsilon(z,0)}_{\text{cubic}},$$

Use scaled variables:

$$\begin{aligned}
\partial_\xi z_{1,1} &= z_{2,1},\\
\partial_\xi z_{2,1} &= C_1 z_{1,1} - C_2 z_{1,1}(z_{1,1}^2 + z_{1,2}^2) + O(\varepsilon),\\
\partial_\xi z_{1,2} &= z_{2,2},\\
\partial_\xi z_{2,2} &= C_1 z_{1,2} - C_2 z_{1,2}(z_{1,1}^2 + z_{1,2}^2) + O(\varepsilon)
\end{aligned}$$

For $\varepsilon = 0$ we have a two-dimensional invariant plane containing two symmetric homoclinics:



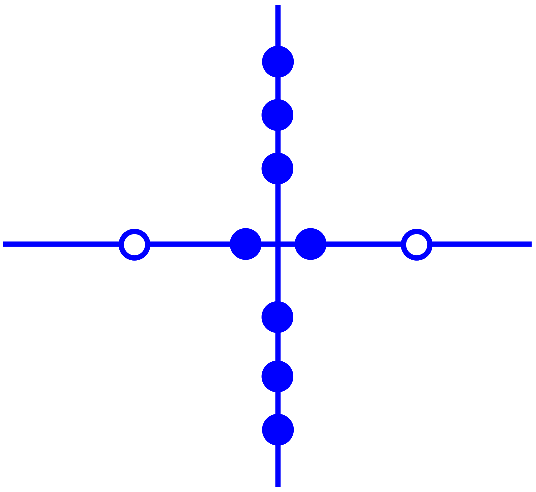
Two symmetric homoclinics p^ε persist for $\varepsilon > 0$ and are estimated by

$$|p^\varepsilon(\xi)| \le c\varepsilon e^{-\varepsilon|\xi|}$$

A NONEXISTENCE RESULT

The homoclinics found in $\{q = 0\}$ generically do not persist for $h^{\varepsilon}(z) \neq 0$

- Dynamical systems arguments:



Three-dimensional stable and unstable manifolds

The manifolds generically do not intersect in an infinite-dimensional phase space

- Global existence theory for one-dimensional quadratic, quasilinear wave equations?

Construct a change of variable (“normal-form transformation”) which

- preserves the structure
- yields the estimate

$$\|h^{\varepsilon}(z)\| \leq c\varepsilon e^{-c^{\star}/\sqrt{\varepsilon}}$$

in the new variables

EXISTENCE THEORY

We look for solutions of the form

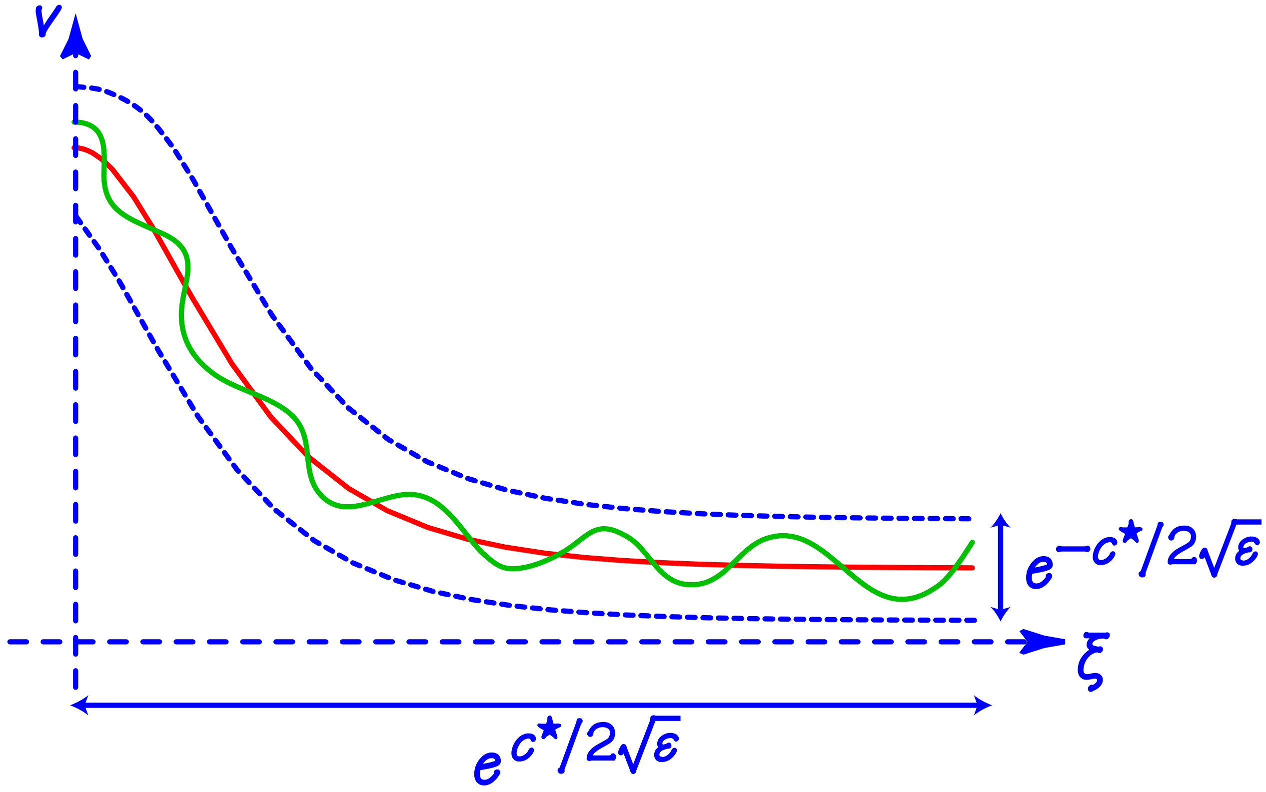
$$(z, q) = (p^{\varepsilon} + Z, q), \quad (Z, q) \in C([0, e^{c^\star/2\sqrt{\varepsilon}}], \mathcal{X}^{s+1})$$

with

$$\|(Z(\xi), q(\xi))\| \leq e^{-c^\star/2\sqrt{\varepsilon}}$$

Combine

- Dynamical-systems arguments
- Kato's iteration scheme



"Centre-stable manifold":

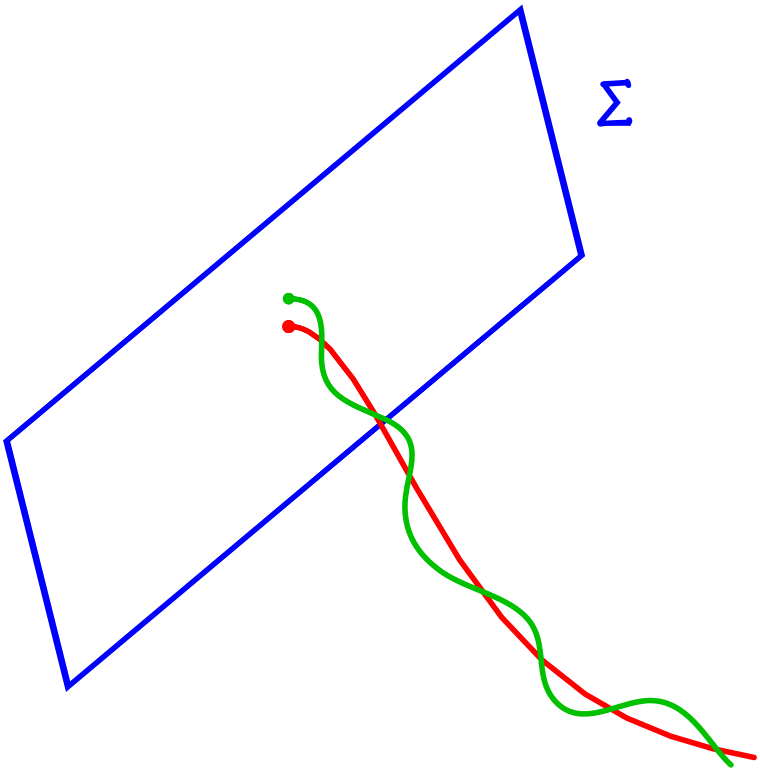
- Define $W^{cs} = \{v(0) \text{ for such solutions } v\}$
- W^{cs} is given as a graph $v_u(0) = f(v_s(0), v_c(0))$

SYMMETRIC PULSES

- Define the “symmetric section” $\Sigma = \mathrm{Fix}\, S$
- A solution $v(\xi)$ on $[0, e^{c^\star/2\sqrt{\varepsilon}}]$ with $v(0) \in \Sigma$ can be extended to a symmetric solution

$$\tilde{v}(\xi) = \begin{cases} v(\xi), & \xi \geq 0 \\ Sv(-\xi), & \xi < 0 \end{cases}$$

on $[-e^{c^\star/2\sqrt{\varepsilon}}, e^{c^\star/2\sqrt{\varepsilon}}]$.



- Solve

$$(I - S)v(0) = 0, \qquad v(0) \in W^{cs}$$

perturbatively around $p^{\varepsilon}(0)$.

- W^{cs} intersects Σ in an infinite family of points (parameterised by $P_c(I - S)v(0)$)