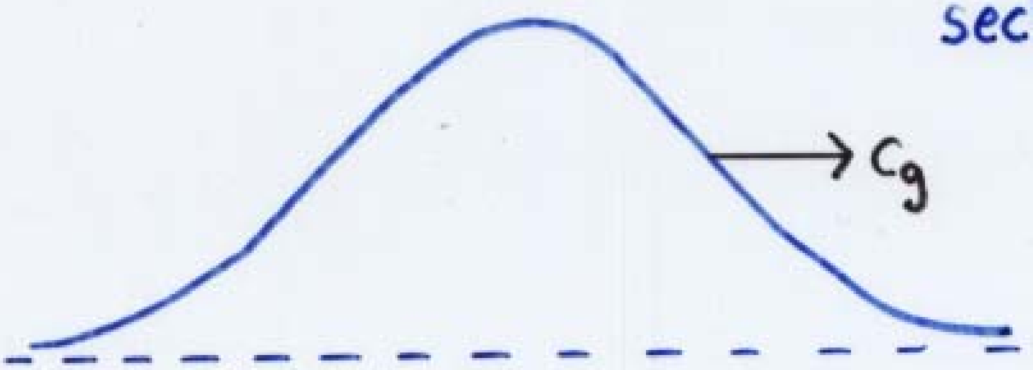
MODULATING PULSES

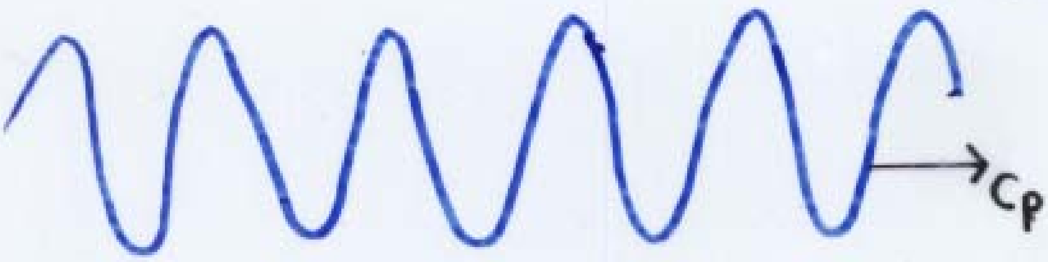
A pulse:

$\operatorname{sech}(x - c_g t)$



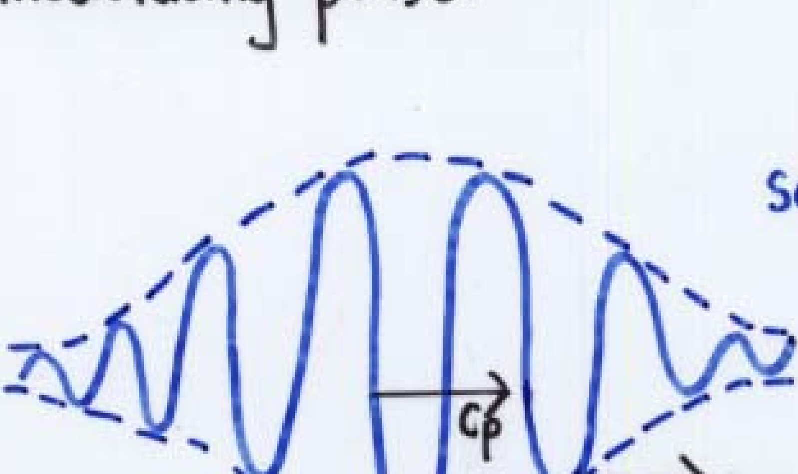
A wavetrain:

$\sin k(x - c_p t)$



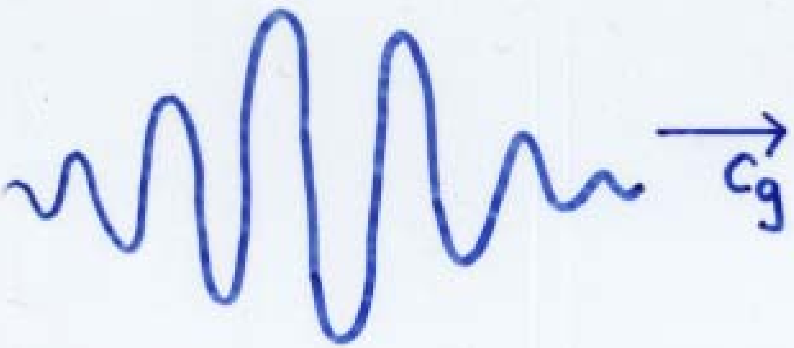
A modulating pulse:

$\operatorname{sech}(x - c_g t) \sin k(x - c_p t)$

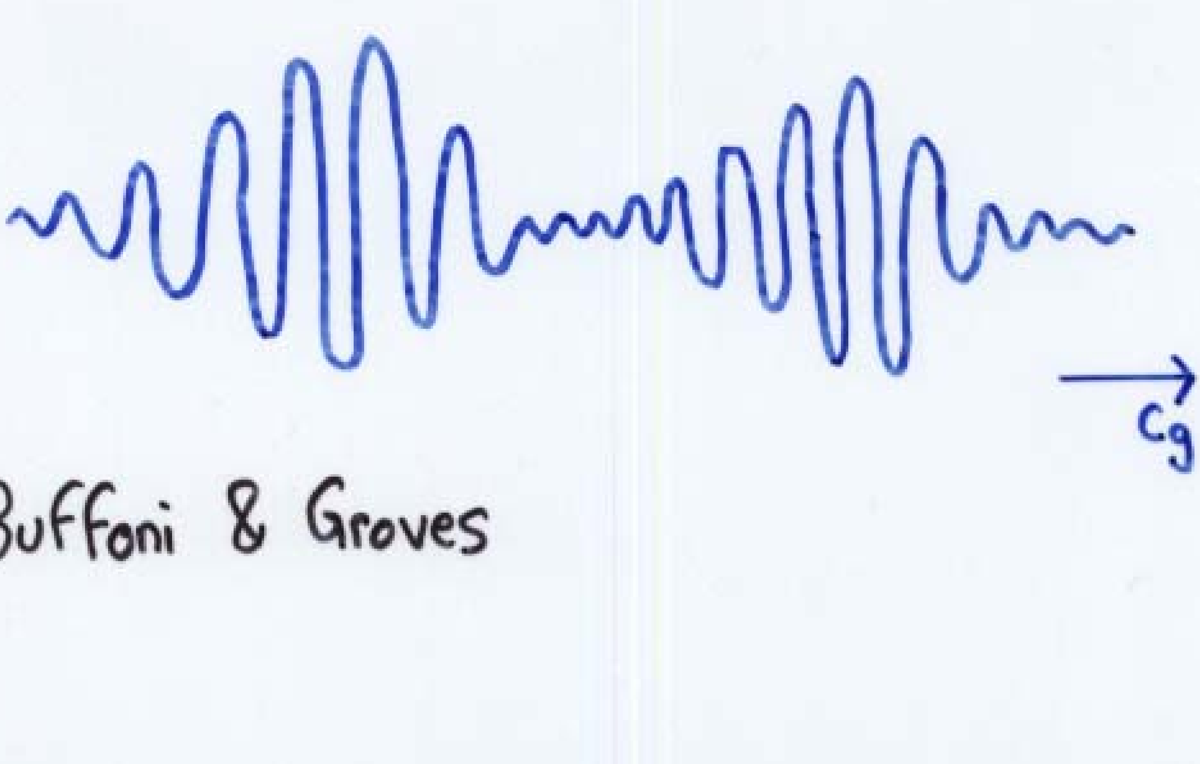


SOLITARY WAVES

Modulating pulses with $c_p = c_g$ are called (generalised) solitary waves.



Iooss & Kirchgässner



Buffoni & Groves

WAVE EQUATIONS

$$u_{tt} = u_{xx} - u + \begin{cases} u^3 \\ -\sin u + u \\ \sinh u + u \end{cases}$$

Linear dispersion relation for $\sin k(x - c_p' t)$:

$$c_p' = (1+k^2)^{1/2}/k$$

$$c_g = \frac{d}{dk}(k c_p') = \frac{1}{c_p'}$$

We look for modulating pulse solutions of the form

$$u(x,t) = v(x - c_g t, x - c_p t)$$

$$\text{"}\operatorname{sech}(x - c_g t) \sin k(x - c_p t)\text{"}$$

where

- $v(\xi, p)$ is periodic in p
- $c_p = c_p' + \gamma_1 \varepsilon^2, \qquad 0 \le \varepsilon \ll 1$
- $c_g = \dfrac{1}{c_p}$

We find that

$$(1 - c_g^2) v_{\xi\xi} + (1 - c_p^2) v_{pp} - v + v^3 = 0$$

SPATIAL DYNAMICS

$$(1-c_g^2)v_{\xi\xi} + (1-c_p^2)v_{pp} - v + v^3 = 0$$

ξ is unbounded $\quad$ p is bounded

We formulate the equation as a system

$$v_\xi = w$$

$$w_\xi = \frac{-(1-c_p^2)}{(1-c_g^2)}v_{pp} + \frac{1}{(1-c_g^2)}(v-v^3)$$

This procedure yields a semilinear evolutionary equation with ξ as time:

$$u_\xi = Lu + N(u), \qquad u=(v,w)\in X$$

Function spaces:

$$X = H^{s+1}_{per} \times H^{s}_{per}, \qquad \mathcal{D}(L) = H^{s+2}_{per} \times H^{s+1}_{per}, \quad s \geq 0$$

Reversibility:

$$\xi \mapsto -\xi, \qquad (v,w) \mapsto (v,-w)$$

Hamiltonian structure:

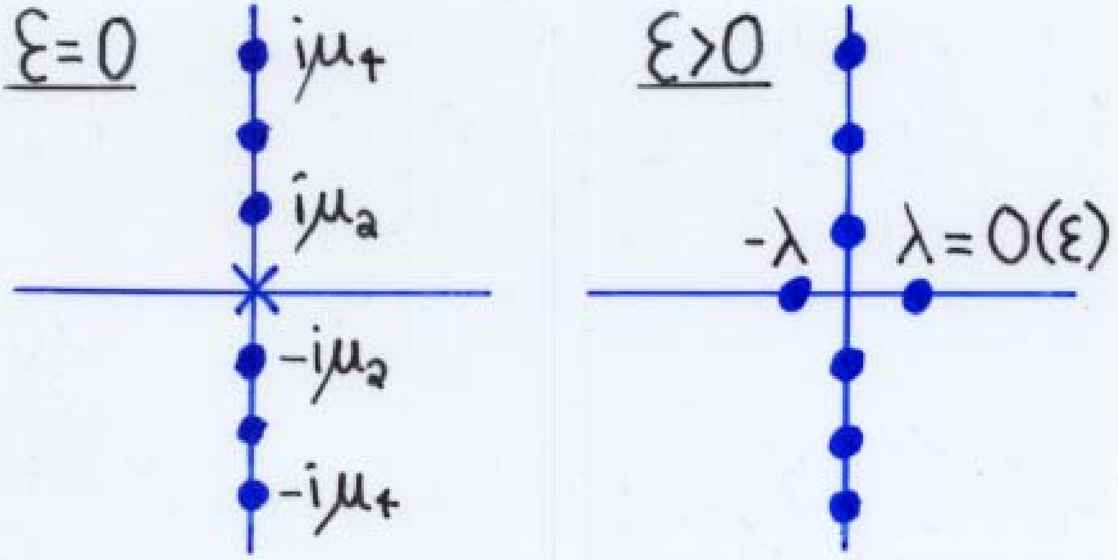
$$v_\xi = \frac{\delta H}{\delta w}, \qquad w_\xi = -\frac{\delta H}{\delta v}$$

$$H = \int\left\{\frac{w^2}{2} - \frac{(1-c_p^2)}{1-c_g^2}v_p^2 - \frac{v^2}{2(1-c_g^2)} + \frac{v^4}{4(1-c_g^2)}\right\}dp$$

FORMULATION AS A DYNAMICAL SYSTEM

$$u_\xi = Lu + N(u) \qquad (*)$$

Spectrum of L:



Expand v,w in sine series:

$$v=\sum_{m=1}^{\infty}\left(\frac{k}{\pi\mu_m}\right)^{1/2} q_m \sin(kmp), \quad w=\sum_{m=1}^{\infty}\left(\frac{k\mu_m}{\pi}\right)^{1/2} p_m \sin(kmp)$$

We obtain a Hamiltonian system with infinitely many degrees of freedom:

$$\partial_\xi q_m = \frac{\partial H}{\partial p_m}, \quad \partial_\xi p_m = -\frac{\partial H}{\partial q_m}, \quad m=1,2,\ldots$$

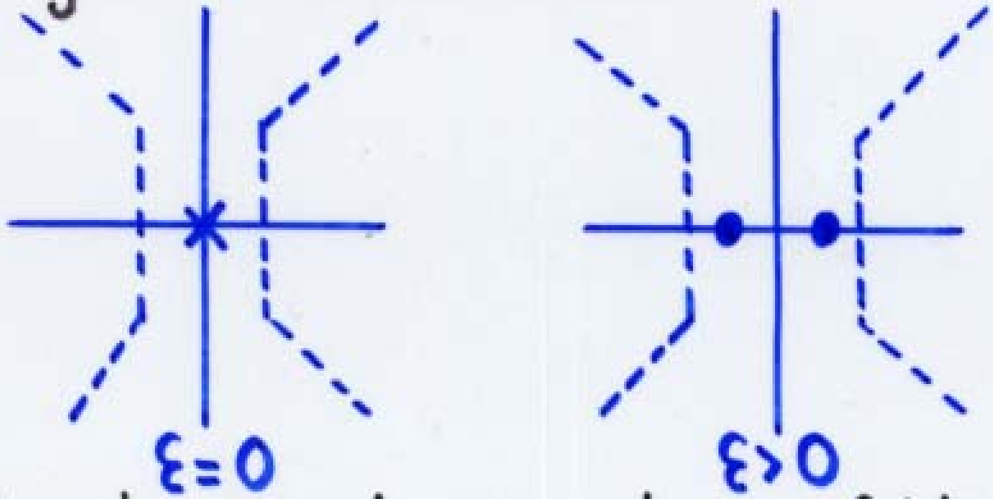
$$H = \frac{p_1^2}{2} - \frac{1}{2}c_1\varepsilon^2 q_1^2 + \sum_{m=2}^{\infty}\frac{\mu_m}{2}(q_m^2+p_m^2) + \ldots$$

This is still the semilinear PDE $(*)$ with

$$u=(q_m,p_m)\in X=(\ell^{s+1/2})^2, \quad \mathcal{D}(L)=(\ell^{s+3/2})^2$$

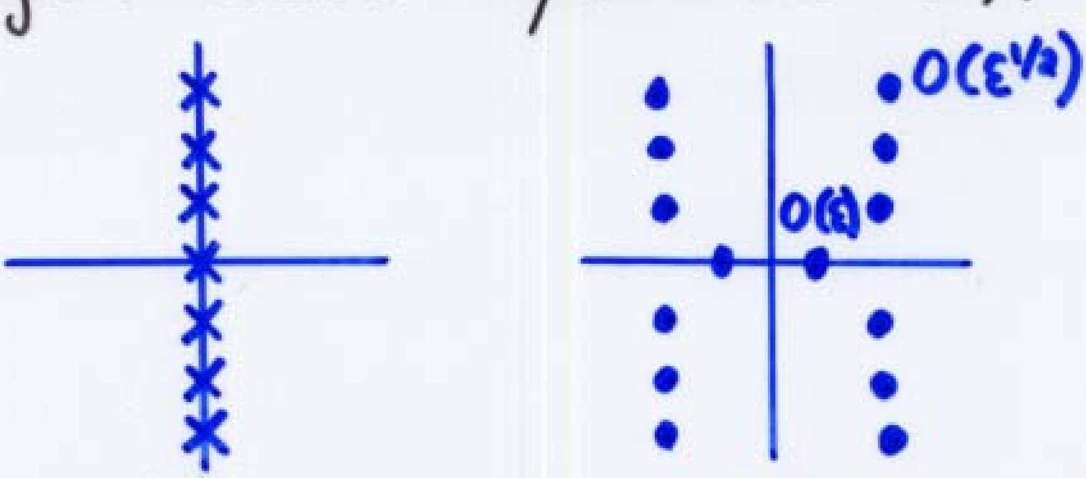
OTHER SITUATIONS

- Kirchgässner (water waves):

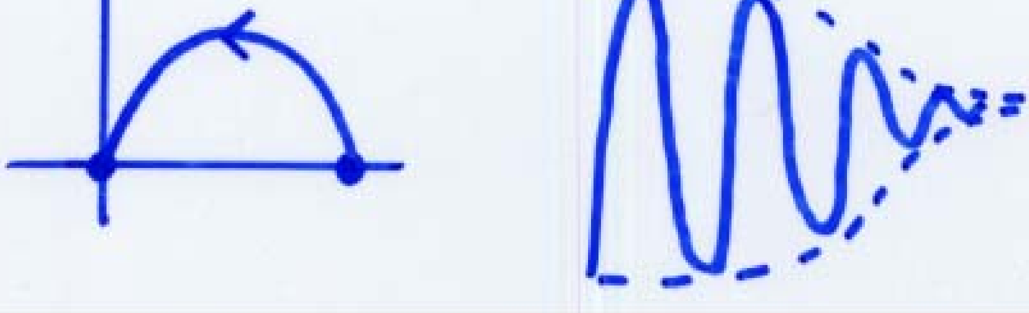


A two-dimensional invariant manifold:

- Haragus & Schneider (Taylor-Couette flow):

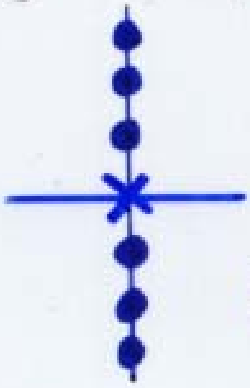


A two-dimensional invariant manifold:

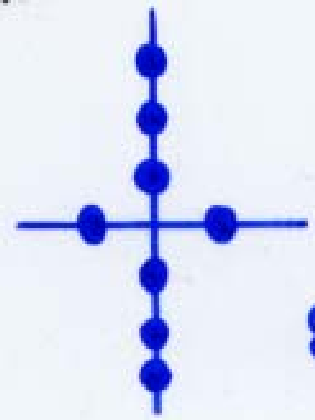


A MODEL PROBLEM

Keep just $(q_1, p_1), \dots, (q_n, p_n)$:



$\varepsilon = 0$



$\varepsilon > 0$

Birkhoff normal form:

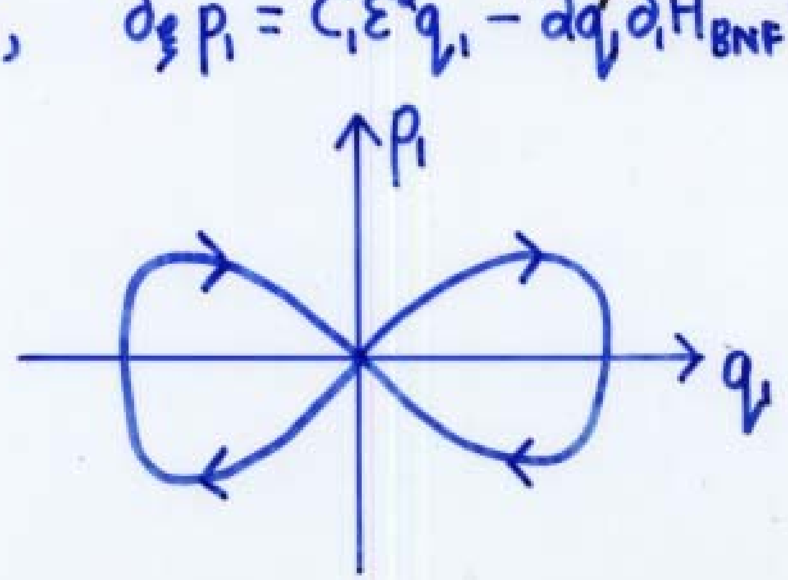
$$H = \frac{1}{2} p_1^2 - \frac{1}{2} c_1 \varepsilon^2 q_1^2 + \sum_{m=2}^{n} \frac{\mu_m}{2} (q_m^2 + p_m^2)$$
$$+ H_{BNF}(q_1^2, q_2^2 + p_2^2, \dots, q_m^2 + p_m^2) + O(N+2)$$

Neglect the $O(N+2)$ remainder terms:

- $q_2^2 + p_2^2, \dots, q_m^2 + p_m^2$ are first integrals
- $S = \{(q_m, p_m) = (0,0), m \geq 2\}$ is invariant

The flow in S is described by

$$\partial_\xi q_1 = p_1, \quad \partial_\xi p_1 = c_1 \varepsilon^2 q_1 - 2 q_1 \partial_1 H_{BNF}(q_1^2, 0, \dots, 0)$$



- Generic nonpersistence upon reinstating remainder terms

NORMAL-FORM THEORY

We use a sequence of symplectic transformations to remove certain terms in the Hamiltonian:

$$\partial_\xi q_m = \frac{\partial H}{\partial p_m}, \quad \partial_\xi p_m = -\frac{\partial H}{\partial q_m}, \quad m = 1,2,3,\ldots$$

$$H = \frac{1}{2} p_1^2 - \frac{1}{2} c_1 \varepsilon^2 q_1^2 + \sum_{m=2}^{\infty} \frac{\mu_m}{2}(q_m^2 + p_m^2) + H_{NF}^{\varepsilon}(q,p) + O(N+2)$$

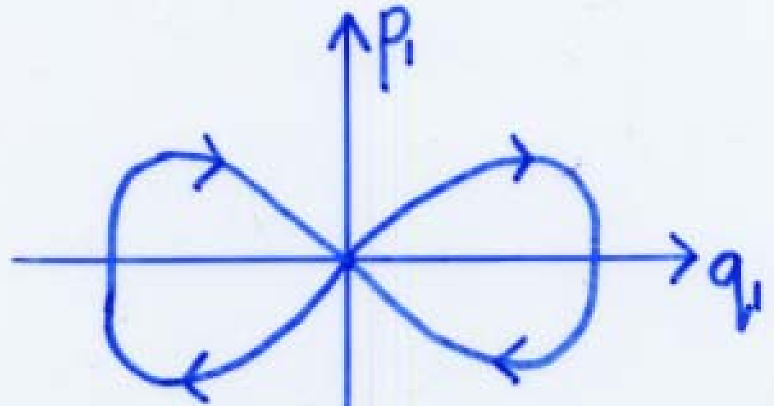
- We cannot remove all 'resonant terms'

Neglect the $O(N+2)$ remainder terms:

- $S = \{(q_1, p_1)\}$ is an invariant subspace
- $H_{NF}^{\varepsilon}\big|_S = F^{\varepsilon}(q_1^2)$

The flow in S is described by

$$\partial_\xi q_1 = p_1, \quad \partial_\xi p_1 = c_1 \varepsilon^2 q_1 - 2 q_1 \partial F^{\varepsilon}(q_1^2)$$



- Generic nonpersistence upon reinstating remainder terms

HALF PULSES

The scaling

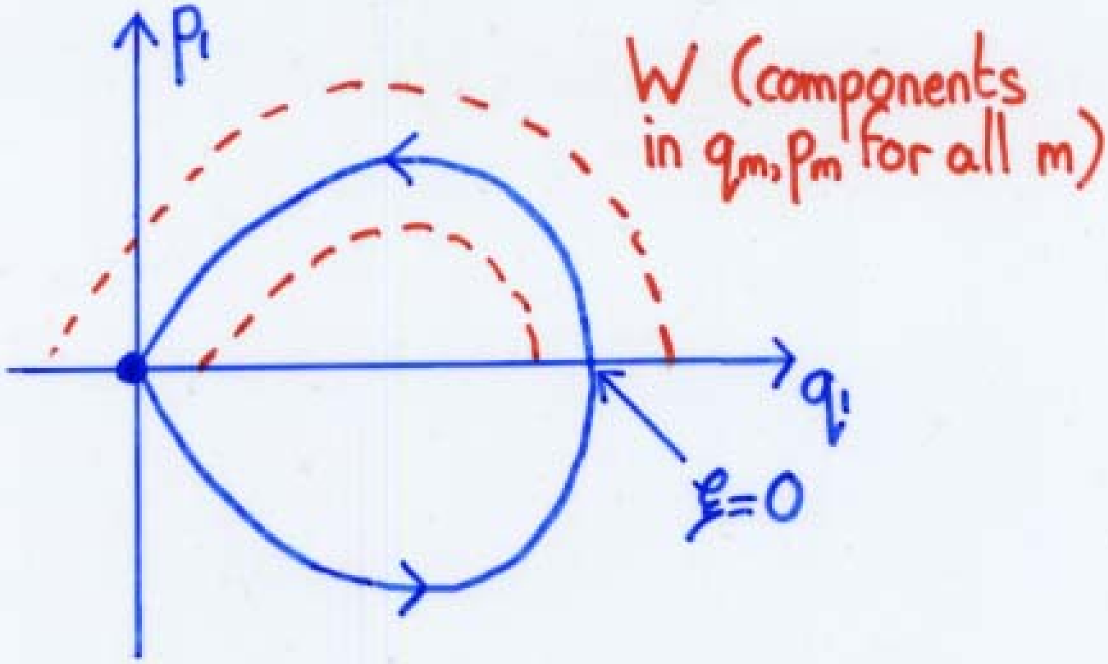
$$u = \varepsilon \hat{u}$$

transforms the equation to

$$u_\xi = \underbrace{N(u)}_{\substack{\text{part in}\\ \text{normal form}}} + \varepsilon^{N+1} \underbrace{R(u)}_{\substack{\text{remainder}\\ \text{terms}}}$$

The normal-form system has a homoclinic solution $h(\xi)$. For the full system we construct a tubular manifold of solutions around h:

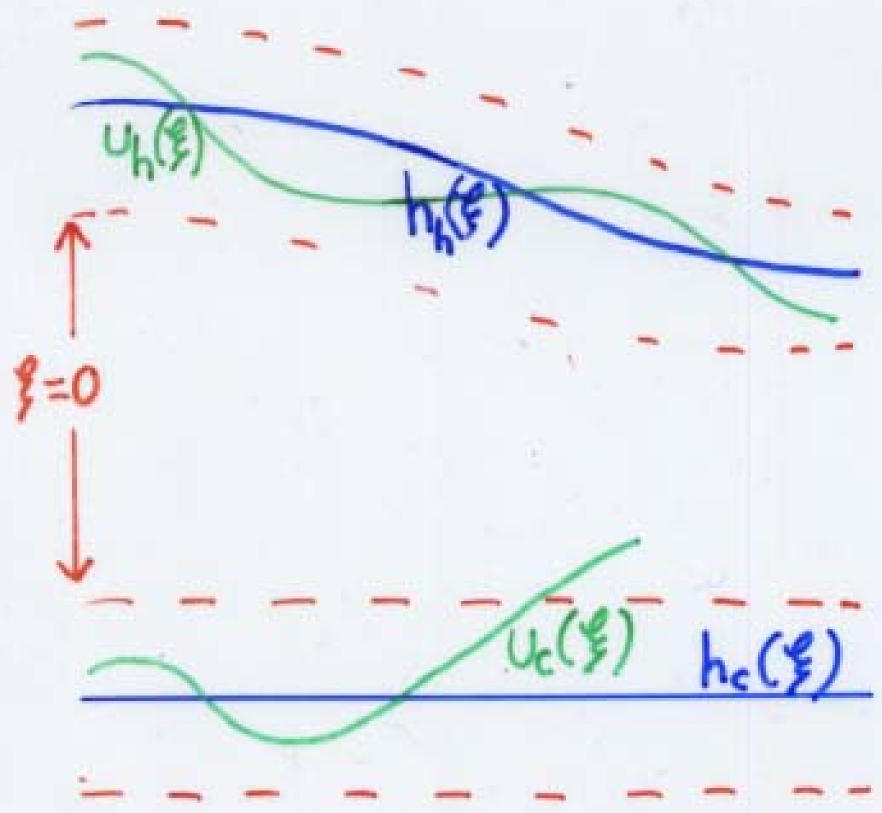
$$W = \{ u(\xi) : \begin{array}{l} u(\xi) \text{ solves } (*) \text{ on } [0,\infty) \\ |u(\xi) - h(\xi)| \leq \varepsilon^n \text{ on } [0,\infty) \end{array} \}$$



- Some of the half pulses making up W can be extended to symmetric pulses

A CENTRE-STABLE MANIFOLD

Consider solutions u on $[0,\infty)$ with the following properties:



The hyperbolic parts of u and h remain $O(\varepsilon^n)$ close for all $\xi \geq 0$

The centre parts of u and h remain $O(\varepsilon^n)$ close for some finite time ξ_0

Define

$$W^{cs} = \{u(0) \text{ for such solutions } u\}$$

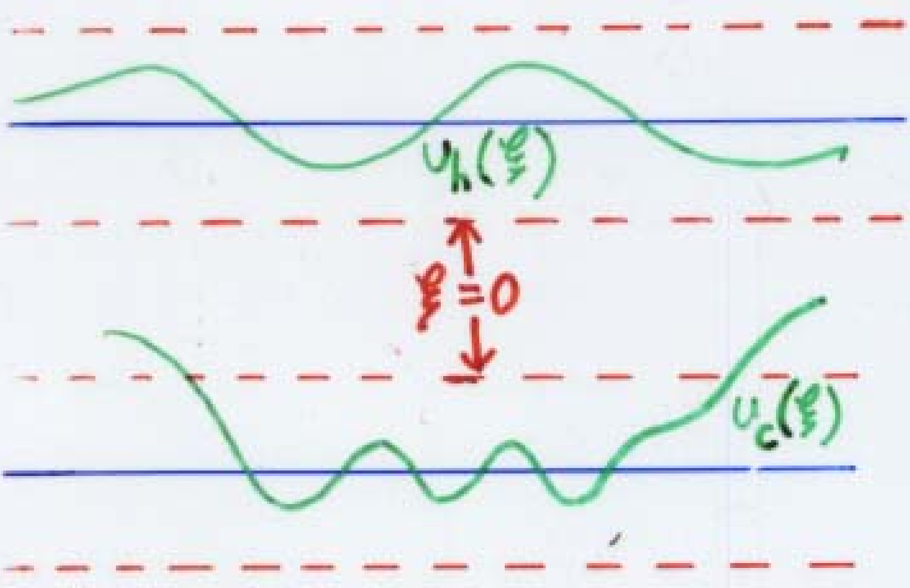
- W^{cs} is parameterised by $u_s(0)$, $u_c(0)$
- W^{cs} is given as a graph

$$u_u(0) = f(u_s(0), u_c(0))$$

<u>Lemma</u>: $\xi_0 = O(1/\varepsilon^{n+1})$

A CENTRE MANIFOLD

Consider solutions u on $\mathbb{R}$ with the following properties:



The hyperbolic part of u remains $O(\varepsilon^n)$ for all $\xi \in \mathbb{R}$

The centre part of u remains $O(\varepsilon^n)$ for some finite negative and positive time

Define

$$W^c = \{u(0) \text{ for such solutions } u\}$$

- W^c is a graph $u_h(0) = \Psi(u_c(0))$

Autonomy:

$$W^c = \{(\Psi(x_c), x_c) : |x_c| \leq \varepsilon^n\}$$

- $$H = \sum_{m=2}^{\infty} \frac{\mu_m}{2}(q_m^2 + p_m^2) + O(|x_h|^2) + O(\varepsilon|x|^2)$$
  $$\geq M|x_c|^2 \quad \text{on } W^c$$

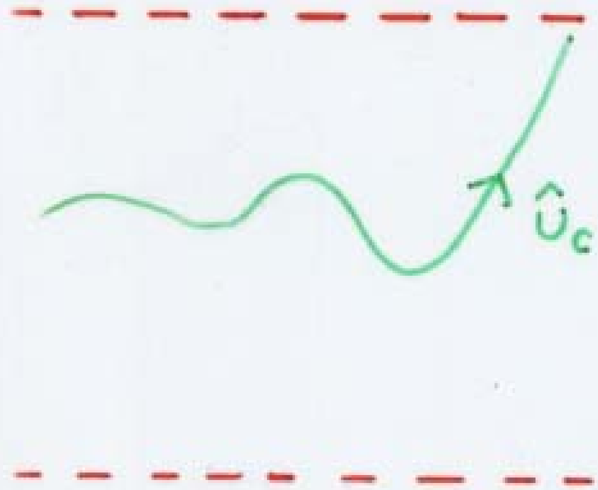
- $|x_c(\xi^*)| \leq M\varepsilon^n \Rightarrow |x_c(\xi)| \leq M\varepsilon^n, \quad \xi \geq \xi^*$

A GLOBAL CENTRE-STABLE MANIFOLD

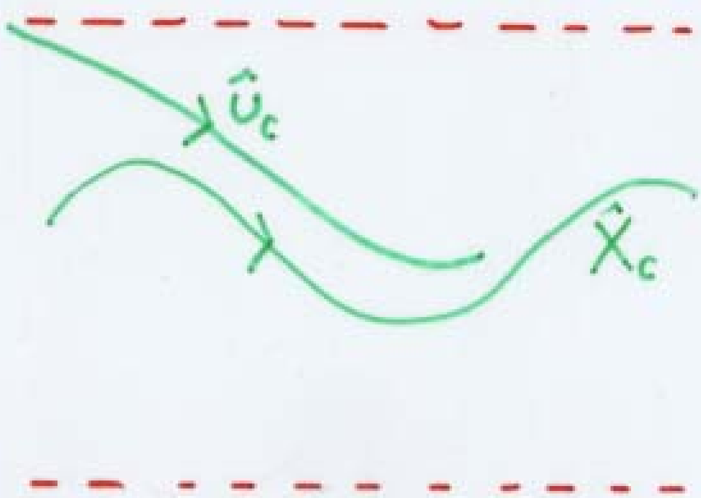
Let $\hat{u}$ be a solution on $[0,\infty)$ with $\hat{u}(0) \in W^{cs}$.

- There exists a solution $\hat{X}$ on W^{c} such that

$$|\hat{u}(\xi) - \hat{X}(\xi)| \leq M_{\varepsilon} \exp(-\lambda \xi), \quad \xi \geq 0$$



- $\hat{u}_c$ remains $O(\varepsilon^n)$ on $O(1/\varepsilon^{n+1})$ timescales



- $\hat{u}_c$ converges exponentially to $\hat{X}_c$
- $\hat{X}_c$ is $O(\varepsilon^n)$ for all ξ

<u>Lemma</u>

- $\hat{u}_c(\xi)$ is $O(\varepsilon^n)$ for all time
- $|\hat{u}(\xi) - \hat{X}(\xi)| = O(\varepsilon^n)$ for all time

SYMMETRIC PULSES

We exploit the reversibility

$$\xi \mapsto -\xi, \qquad (q,p) \mapsto (q,-p)$$

We seek intersections of

W^{cs} — manifold of initial data for solutions u with $|u(\xi)-h(\xi)| \le \varepsilon^n$, $\xi \geq 0$

Π — the symmetric section $\{p=0\}$

We write our equation as

$$\hat{u}_\xi = N(\hat{u}) + \cancel{\varepsilon^{N+1}}\rho R(\hat{u})$$

and consider

$$[\hat{u}_{cs}(0)]_{p=0} = 0$$

as

$$\mathcal{F}(\underbrace{v_s, v_c}_{\substack{\text{stable/centre} \\ \text{parts of } \hat{u}_{cs}}}, \rho) = 0, \qquad \mathcal{F}(0,0,0) = 0$$

Implicit function theorem:

$$v_s = v_s(v_c, \rho)$$