DIMENSION BREAKING

Solutions of the KP equation

$$\partial_{xx}\left(u_{xx} - u - \frac{3}{2}u^2\right) - u_{zz} = 0 \qquad \text{(KP)}$$

which do not depend upon z and decay as $x \to \pm\infty$ satisfy the KdV equation

$$u_{xx} - u - \frac{3}{2}u^2 = 0 \qquad \text{(KdV)}$$

(KdV) has the solitary-wave solution

$$u(x) = -\operatorname{sech}^2\left(\frac{x}{2}\right)$$

(KP) has a family

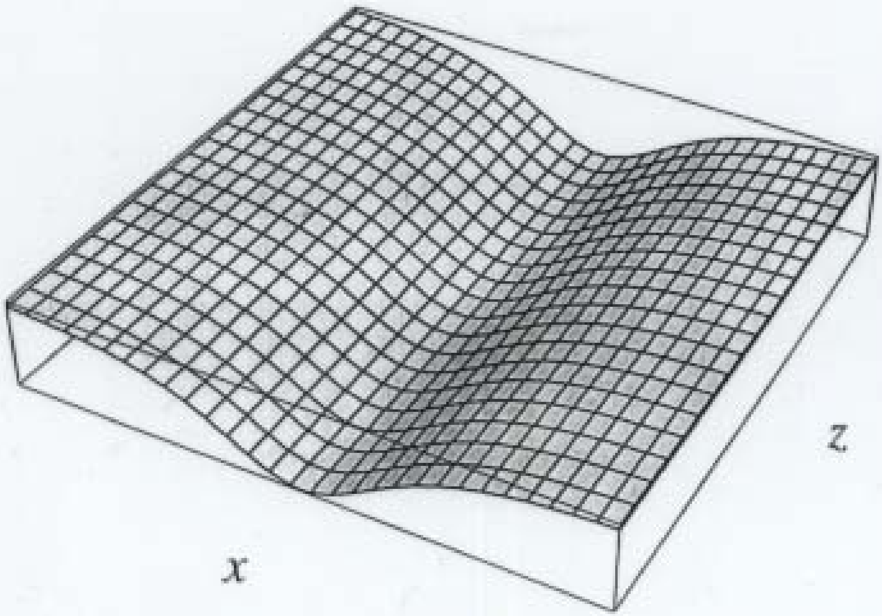
$$u_\varepsilon(x,z) = -\frac{4(1-\varepsilon^2)}{4-\varepsilon^2}\,\frac{1-\varepsilon\cosh(a_\varepsilon x)\cos(\omega_\varepsilon z)}{(\cosh(a_\varepsilon x) - \varepsilon\cos(\omega_\varepsilon z))^2}, \quad \varepsilon \in [0,1)$$

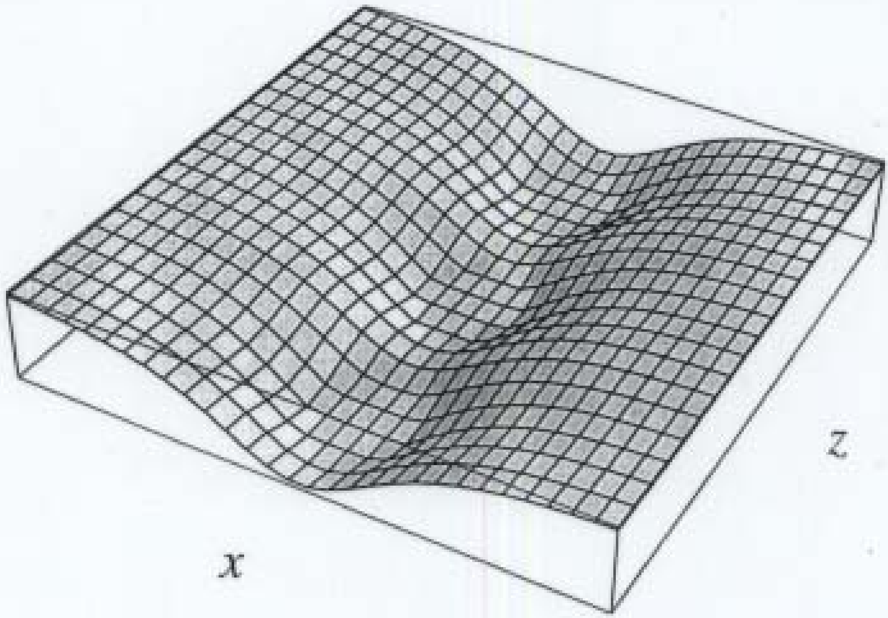
$$a_\varepsilon = \left(\frac{1-\varepsilon^2}{4-\varepsilon^2}\right)^{1/2}, \qquad \omega_\varepsilon = \frac{(3(1-\varepsilon^2))^{1/2}}{4-\varepsilon^2}$$

of waves which are periodic in z and decay as $x \to \pm\infty$.

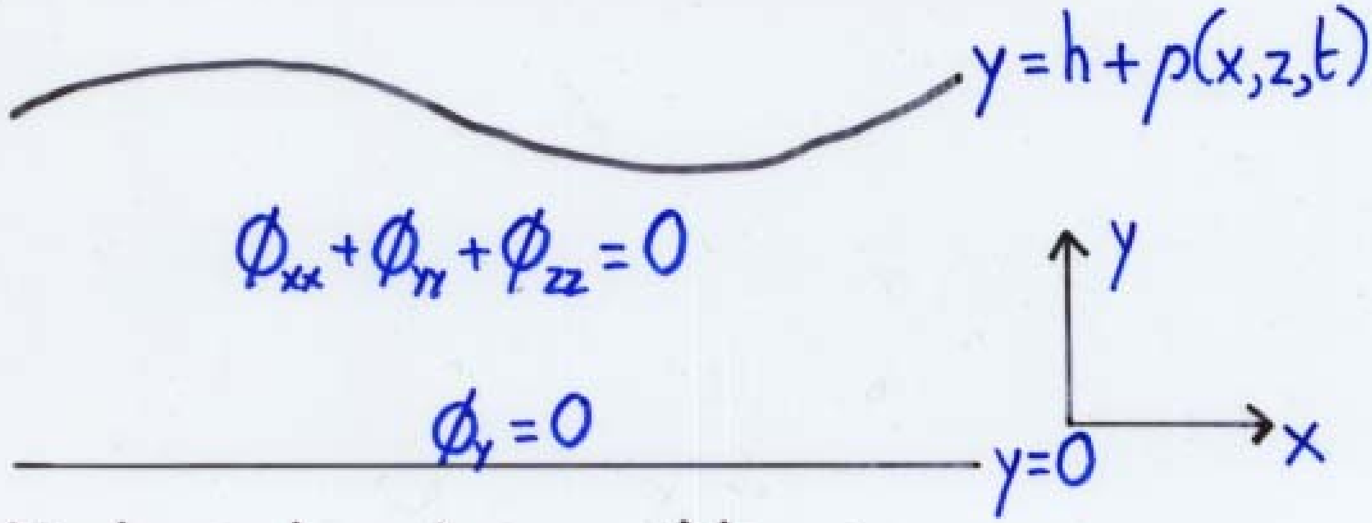
- u_0 is the KdV <u>line solitary wave</u>
- u_ε, $\varepsilon \in (0,1)$ is a <u>periodically modulated solitary wave</u>
- u_1 is a <u>fully localised solitary wave</u> (it decays as $z \to \pm\infty$)

The periodically modulated solitary waves emerge from the line solitary wave in a <u>dimension-breaking bifurcation</u>.


z
x


z
x

THE WATER-WAVE PROBLEM



$$\phi_{xx}+\phi_{yy}+\phi_{zz}=0$$

$$\phi_y=0$$

Nonlinear boundary conditions:

$$\rho_t=\phi_y-\rho_x\phi_x-\rho_z\phi_z$$

$$\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)+g\rho$$
$$-\sigma\left[\frac{\rho_x}{\sqrt{1+\rho_x^2+\rho_z^2}}\right]_x-\sigma\left[\frac{\rho_z}{\sqrt{1+\rho_x^2+\rho_z^2}}\right]_z=0$$

Travelling waves:

$$\rho(x,z,t)=\rho(x-ct,z) \qquad \phi(x,y,z,t)=\phi(x-ct,y,z)$$

Parameters: $\alpha=gh/c^2$, $\beta=\sigma/hc^2$

- Line solitary waves: $\rho(x,z)=\rho(x)\to 0$ as $x\to\pm\infty$
- Periodically modulated solitary waves:

$$\rho(x,z+P)=\rho(x,z)$$
$$\rho(x,z)\to 0 \quad \text{as} \quad x\to\pm\infty$$

MODEL EQUATIONS

KdV equation

$$u_{xx} - u - \frac{3}{2}u^2 = 0$$

Formulated 1895 as a model equation for 2D water waves

Line solitary wave

$$u = -\operatorname{sech}^2\left(\frac{x}{2}\right)$$

Dimension-breaking bifurcation

KP equation

$$\partial_{xx}\left(u_{xx} - u - \frac{3}{2}u^2\right) - u_{zz} = 0$$

Formulated 1970 as a model equation for 3D water waves

Periodically modulated solitary waves

Euler equations for 2D water waves

Amick & Kirchgässner (1989): line solitary wave

$$\eta = -\varepsilon \operatorname{sech}^2\left(\frac{\varepsilon^{1/2} x}{2(\beta - 1/3)^{1/2}}\right) + O(\varepsilon^2)$$

$$\alpha = 1 + \varepsilon, \quad \beta > 1/3$$

Dimension-breaking bifurcation?

Euler equations for 3D water waves

Periodically modulated solitary waves?

SPATIAL DYNAMICS

Formulate the 3D water-wave problem as an evolutionary equation

$$u_z = L_\varepsilon u + N_\varepsilon(u), \qquad u \in X \qquad (1)$$

z is the 'time-like' variable

X is a Hilbert space of functions which decay as $x \to \pm\infty$

Observation:

Equilibrium solutions of (1) $\longleftrightarrow$ Line solitary waves

Write

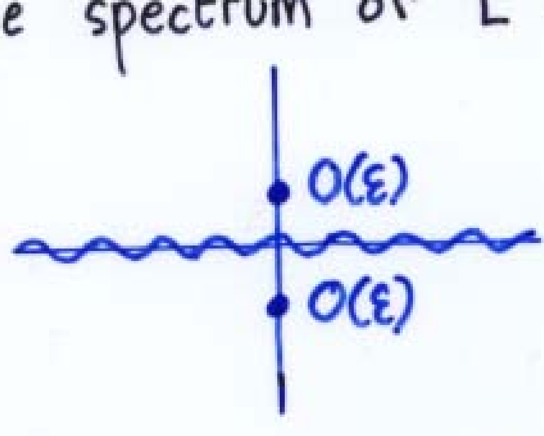
$$u(x,z) = u^*(x) + w(x,z)$$

Line solitary wave (Amick & Kirchgässner)

and study

$$w_z = Lw + N(w) \qquad (2)$$

The spectrum of L for $\alpha \in (1, 1+\varepsilon)$, $\beta > 1/3$:



- Lyapunov centre theorem $\Rightarrow$ periodic solutions of (2)
- Periodic solutions of (2) $\Rightarrow$ Periodically modulated solitary waves

EVOLUTIONARY EQUATIONS

Formal variational principle:

$$\delta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_0^{1+\rho(x,z)}\left(-\phi_x + \tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)\right)dy + \tfrac{1}{2}\alpha\rho^2 + \beta\left[\sqrt{1+\rho_x^2+\rho_z^2}-1\right]\right\}dx\,dz = 0$$

New variables:

$$\tilde{y} = y/(1+\rho), \qquad \Phi(x,\tilde{y},z) = \phi(x,y,z)$$

$$\Rightarrow \delta\mathcal{L} = 0, \qquad \mathcal{L} = \int_{-\infty}^{\infty} L(\rho,\Phi,\rho_z,\Phi_z)\,dz$$

Legendre transform:

$$\omega = \frac{\delta\mathcal{L}}{\delta\rho_z}, \qquad \xi = \frac{\delta\mathcal{L}}{\delta\Phi_z}$$

$$\Rightarrow \rho_z = \rho_z(\rho,\omega,\Phi,\xi), \qquad \Phi_z = \Phi_z(\rho,\omega,\Phi,\xi)$$

Hamiltonian:

$$H(\rho,\omega,\Phi,\xi) = \int_{-\infty}^{\infty}\int_0^1 \Phi_z\xi\,d\tilde{y}\,dz + \int_{-\infty}^{\infty}\rho_z\omega\,dz - L(\rho,\Phi,\rho_z,\Phi_z)$$

Hamilton's equations:

$$\rho_z = \frac{\delta H}{\delta\omega}, \quad \omega_z = -\frac{\delta H}{\delta\rho}, \quad \Phi_z = \frac{\delta H}{\delta\xi}, \quad \xi_z = -\frac{\delta H}{\delta\Phi}$$

+ nonlinear boundary conditions

↑

Eliminate with a 'linearising diffeomorphism'

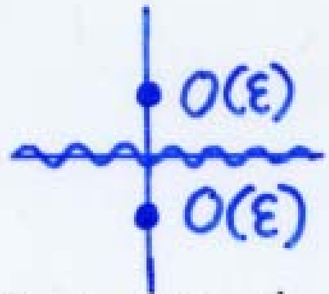
EVOLUTIONARY EQUATIONS

$$\frac{d}{dz}\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix} = L_c\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix} + N_c\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix}$$

- We study this evolutionary equation in a Hilbert space of functions which decay as $x \to \pm\infty$.
- Equilibrium solutions of (1) $\leftrightarrow$ Line solitary waves
- Seek solutions

$$u(x,z) = \underset{\uparrow\ \text{Line solitary wave}}{u^*(x)} + w(x,z)$$

$$w_z = Lw + N(w)$$

$\sigma(L)$: 

- λ is a spectral point of L if $(L-\lambda I)^{-1}$ does not exist
  - λ is an eigenvalue if $L-\lambda I$ is not injective
  - Otherwise λ is a point of the essential spectrum of L

LINEAR SPECTRAL ANALYSIS

Scaled resolvent equations:

$$L\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix} - ik\varepsilon\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix} = \begin{pmatrix}\rho^\dagger\\ \omega^\dagger\\ \Phi^\dagger\\ \xi^\dagger\end{pmatrix} \qquad \begin{matrix}(1)\\ (2)\\ (3)\\ (4)\end{matrix}$$

with boundary conditions $\Phi_{\tilde y}|_{\tilde y=0,1} = B(\rho,\Phi)$.

- Solve (1), (2), (4):

$$\rho=\rho(\Phi_x, k\Phi, w^\dagger), \quad \omega=\omega(k\Phi, w^\dagger), \quad \xi=\xi(k\Phi, w^\dagger)$$

- Insert into (3), BCs, Fourier transform, Green's function:

$$\hat{\Phi} = \int_0^1 \underbrace{G}_{=G_0+\text{h.o.t.}}\, \mathcal{F}(\hat{\Phi}, w^\dagger)\, d\xi + G|_{\xi=0,1}\, \mathcal{F}(\hat{\Phi}, w^\dagger)$$

- Write $\Phi(x,\tilde y) = \Phi_1(x) + \Phi_2(x,\tilde y)$, where

$$\Phi_2 = \Phi_2(\Phi_{1x}, k\Phi_1, w^\dagger)$$

$$\underbrace{\Phi_{1xxxx} - \Phi_{1xx} + 3\left(\operatorname{sech}^2\left(\tfrac{x}{2}\right)\Phi_{1x}\right)_x}_{\text{This operator has } \sigma=[0,\infty)\cup\{-\omega_0^2\}} + k^2\Phi_1 + \varepsilon^{1/4}R_{\varepsilon,k}(\Phi_1) = \mathcal{F}(w^\dagger)$$

- Hence $L - ik\varepsilon I$ is invertible except when

$$k^2 = k_\varepsilon^2, \qquad |k_\varepsilon - \omega_0| = O(\varepsilon^{1/4})$$

LINEAR SPECTRAL ANALYSIS

The origin lies in the 'essential spectrum' of L:

$$L\begin{pmatrix}\rho\\ \omega\\ \Phi\\ \xi\end{pmatrix}=\begin{pmatrix}\rho^{+}\\ \omega^{+}\\ \Phi^{+}\\ \xi^{+}\end{pmatrix} \qquad \xi^{+}=(\xi_1^{+})_x+(\xi_2^{+})_y$$

$$\Phi_{\tilde{y}}=B(\rho,\Phi) \quad \text{on } \tilde{y}=0,1$$

Use the same reduction procedure:

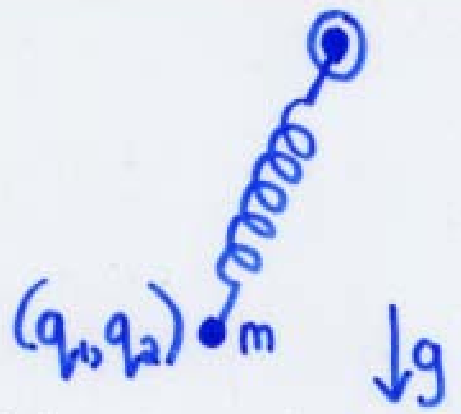
- $\omega=\omega(w^{+}), \quad \xi=\xi(w^{+})$
- $\rho=\rho(\Phi_{1x},\Phi_2,w^{+})$
- $\Phi_2=\Phi_2(\Phi_{1x},w^{+})$

Φ_1 satisfies the equation

$$\Phi_{1xxxx}-\Phi_{1xx}+3\left(\operatorname{sech}^2\left(\tfrac{x}{2}\right)\Phi_{1x}\right)_x+\varepsilon^{1/4}\left(m\left(\Phi_{1x}\right)\right)_x$$
$$=\partial_x\left(\mathcal{F}(\omega^{+})\right)+\partial_x\left(\mathcal{F}(\xi_1^{+})\right)$$

- We integrate once with respect to x
- The operator
  $$\partial_{xx}+3\operatorname{sech}^2\left(\tfrac{x}{2}\right)-1$$
  is invertible. Hence we can determine Φ_{1x} but not Φ.
- The function $N(w)$ belongs to the range of L.

A SIMPLE EXAMPLE



A spring pendulum

- Variational principle:

$$\delta \int L\,dt = 0, \quad L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + mgq_2 - \sigma\left((q_1^2 + q_2^2)^{1/2}\right)$$

- Legendre transform:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1, \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2, \quad H = p_1\dot{q}_1 + p_2\dot{q}_2 - L$$

$$\downarrow$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}$$

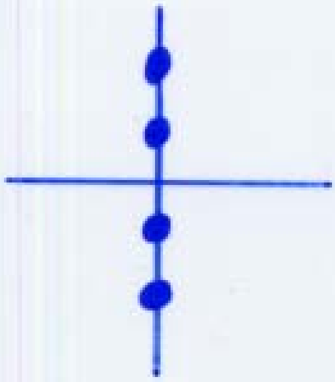
- $(0, -\ell, 0, 0)$ is an equilibrium:

$$u = (0, -\ell, 0, 0) + w$$

$$\downarrow$$

$$\dot{w} = Lw + N(w)$$

$\searrow$ $\sigma(L)$:



$\pm i\sqrt{g/\ell}$

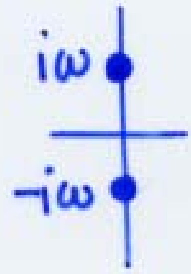
$\pm i\sqrt{\sigma''(\ell)/m}$

LYAPUNOV CENTRE THEOREM

- Classical form for Hamiltonian systems:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j=1,\dots,n$$

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j=1,\dots,n$$



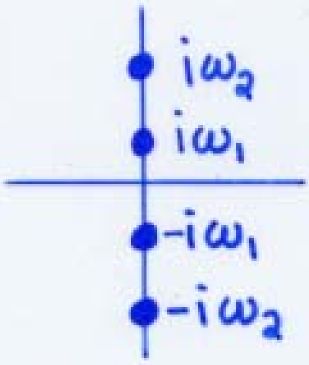
Nonresonance condition: $in\omega$, $n \neq \pm 1$ is not an eigenvalue

$\Rightarrow$ There exists a family of solutions $u=(q,p)$

$$u_s(t+2\pi/\omega_s) = u_s(t), \quad |u_s(t)| \ll 1$$

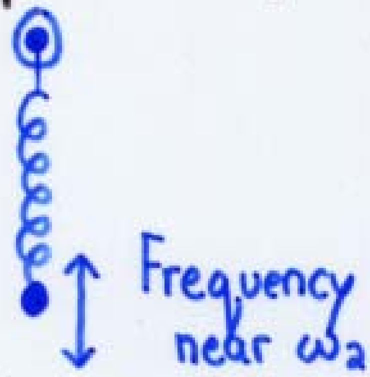
$$\omega_s \to \omega, \quad u_s \to 0 \quad \text{as } s \to 0$$

Example (spring pendulum):

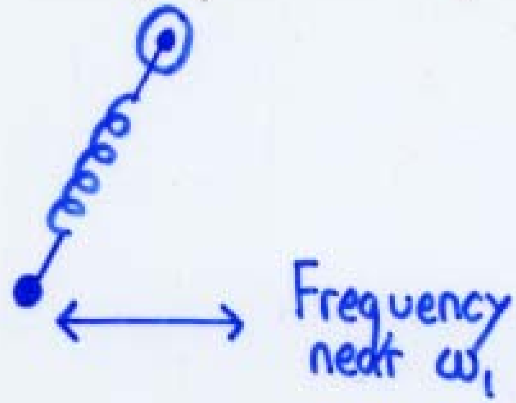


Generically $\omega_2/\omega_1 \in \mathbb{Z}$

First Lyapunov family:



Second Lyapunov family:



GENERALISATIONS

- Moser/Weinstein: The nonresonance condition can be dropped if $H''(0)>0$. For example

$$H=\frac{\omega}{2}(p_1^2+q_1^2)+\omega(p_2^2+q_2^2)+\ldots$$

$\Rightarrow$ Two Lyapunov families (frequencies $\sim\omega$, $\sim 2\omega$)

- Devaney: The Hamiltonian structure can be replaced by reversibility:

$$u_t = Lu + N(u) \qquad u\in X$$

is reversible if u a solution $\Leftrightarrow$ Su a solution, $S^2=I$, $SL=-LS$, $SN=-NS$. Hamiltonian systems:

$$S(q,p)=(q,-p)$$

- Iooss:
  - Devaney's method does not require X to be finite dimensional. We need that $\pm in\omega$, $n\neq\pm1$ is not a spectral point.
  - Devaney's method does not require that 0 is not a spectral point. We need merely that

$$Lu = N(w)$$

has a unique solution u for each w.

PERIODIC SOLUTIONS

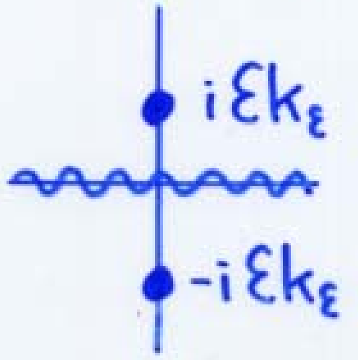
What do we know about our spatial dynamics problem

$$w_z = Lw + N(w), \qquad w \in X \,?$$

- It is a reversible Hamiltonian system

$$S(p,\omega,\Phi,\xi) = (p,-\omega,\Phi,-\xi)$$

- $\sigma(L)$:



- 0 belongs to the essential spectrum of L. The equation

$$Lu = N(w)$$

has a unique solution u for each w, but only the derivatives of Φ are determined

- L and N depend upon Φ only through its derivatives
- The Devaney/Iooss method yields a family

$$w_s(x, z + 2\pi/\omega_s) = w_s(x,z), \quad |w_s| \ll 1,$$
$$\omega_s \to \varepsilon k_\varepsilon, \quad w_s \to 0 \quad \text{as} \quad s \to 0$$

- Periodically modulated solitary waves $u^*(x) + w_s(x,z)$

THE DEVANEY-IOOSS METHOD

$$w_z = Lw + N(w), \qquad w \in X$$

$$Le = i\omega e, \quad L\bar{e} = -i\omega\bar{e}, \quad Se = \bar{e}$$

We write

$$s = (\omega + \mu)z, \qquad w = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} w_n e^{ins}, \quad w_{-n} = \overline{w_n}$$

to arrive at

$$Lw_0 = -[N(w)]_0 \tag{1}$$

$$(L - i\omega I)w_n = -[N(w)]_n + in\mu w_n, \quad n = 2, 3, \ldots \tag{2}$$

and

$$(L - i\omega I)w_{11} = \underbrace{-[N(w)]_1 + i\mu(w_{11} + ae),}_{\text{Solvability condition } \Omega(\ldots, \bar{e}) = 0 \ (4)} \quad w_1 = w_{11} + ae \tag{3}$$

- Project the RHS of (3), solve (1),(2),(3):

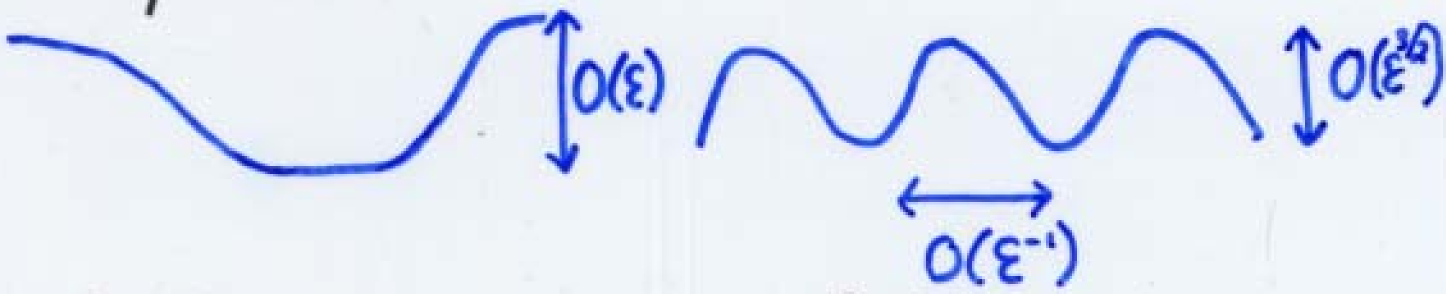
$$w_0 = \mathcal{M}_0(\mu, a, \bar{a}), \quad w_{11} = \mathcal{M}_{11}(\mu, a, \bar{a}), \quad w_n = \mathcal{M}_n(\mu, a, \bar{a})$$

- Substitute $w = \mathcal{M}_0 + (ae + \mathcal{M}_{11})e^{is} + \mathcal{M}_2 e^{2is} + \ldots$ into (4):

$$\underbrace{\mu + g(\mu, |a|^2) = 0}_{\text{Reversibility used to deduce this structure}} \Rightarrow \mu = \Lambda(|a|^2)$$

COMMENTS

- We have constructed periodically modulated solitary waves.



Profile in x　　　　　　Profile in z

We cannot construct fully localised solitary waves by taking limits.

- It is a consequence of the dimension-breaking result that the Amick & Kirchgässner line solitary wave is unstable with respect to perturbations of the type

$$u(x,y,z,t) = e^{\sigma t} \underbrace{U(x,y,z)}, \qquad 0<\sigma<<1$$

We can find U_σ, periodic in z, so that u satisfies the linearised water-wave problem around $u^*(x,y)$