4

PLANE WAVES

The purpose of this chapter is to describe in some detail the character of the plane waves whose interactions we will be considering in the subsequent chapters.

4.1 The class of *pp*-waves

It is first convenient to introduce the widely known class of *pp*-waves. These are plane-fronted gravitational waves with parallel rays. They are defined by the property that they admit a covariantly constant null vector field. It is possible to interpret such a field as the rays of gravitational or other null waves.

It is possible to identify the tetrad vector l^μ with this field. It can then be seen from (2.12) that the defining property that $l_{\mu;\nu} = 0$, among other conditions, immediately implies that $\rho = \sigma = \kappa = 0$. It follows that this vector field is tangent to a non-expanding, shear-free and twist-free null geodesic congruence. Since the congruence is twist-free, there exists a family of 2-surfaces orthogonal to l^μ that may be considered as wave surfaces (Kundt, 1961). Moreover, since the congruence is expansion-free, the wave surfaces are plane and since, in addition, $\tau = 0$, the rays orthogonal to the wave surfaces are parallel.

This class of solutions was first discovered by Brinkmann (1923), and subsequently rediscovered by several authors, for example Peres (1959). Using a null coordinate u, defined such that $l_\mu = u_{,\mu}$, the metric can be written in the Kerr–Schild form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + H(u,X,Y)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.1}$$

where the coordinates X and Y span the wave surfaces. These metrics are either of algebraic type N, or are conformally flat. The only non-zero components of the curvature tensor are given by

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{4}(H_{,XX} + H_{,YY}) \\ \Psi_4 &= \tfrac{1}{4}(H_{,XX} - H_{,YY} + 2iH_{,XY}). \end{aligned} \tag{4.2}$$

For reviews of these solutions see Takeno (1961), Ehlers and Kundt (1962), Zakharov (1973), and Kramer *et al.* (1980).

The gravitational wave can be expressed in terms of the single component $\Psi_4 = Ae^{i\alpha}$. It is then possible to use the analogy between the Weyl tensor component Ψ_4 and the electromagnetic field tensor component Φ_2, and to regard A as the amplitude of the gravitational wave and α as its polarization. In this form, the vacuum *pp*-waves for which α is constant are said to be linearly polarized.

It may also be noticed at this point that the vacuum field equations for space-times with the line element (4.1) reduce to the two-dimensional Laplace equation

$$H_{,XX} + H_{,YY} = 0. \tag{4.3}$$

Since this equation is linear, it follows that distinct solutions with different expressions for H may be simply superposed. It follows that independent gravitational waves of this type which propagate in the same direction do not interact. This was first pointed out by Bonnor (1969) and Aichelburg (1971).

There is another interesting result that has been proved by Yurtsever (1988*b*) which relates to the class of so-called 'sandwich waves'. This states that any gravitational wave space-time that is flat before the arrival of the wave and returns to perfect flatness after the wave passes is necessarily a *pp*-wave space-time.

4.2 The class of plane waves

In Einstein–Maxwell theory, the particular class of *plane waves* are defined to be *pp*-waves in which the field components are the same at every point of the wave surfaces. This is the sense in which they are said to have 'plane symmetry'. This class of solutions was first considered by Baldwin and Jeffery (1926).

Using the above notation, this condition requires that Ψ_4 and Φ_2 are independent of the space-like coordinates X and Y which span the wave surfaces. The line element for a plane wave can thus be written in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + (h_{11}X^2 + 2h_{12}XY + h_{22}Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.4}$$

where h_{ij} are functions of u only. Terms in H that are linear in X and Y have been removed by a simple coordinate transformation. The non-zero curvature tensor components are then

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{2}(h_{11} + h_{22}) \\ \Psi_4 &= \tfrac{1}{2}(h_{11} - h_{22} + 2ih_{12}). \end{aligned} \tag{4.5}$$

The line element (4.4) describes a vacuum space-time, in this case a pure gravitational plane wave, if $h_{22} = -h_{11}$. The gravitational wave will have constant linear polarization if, in addition, h_{12} is proportional to h_{11}. In this case the metric function H can be expressed in the form

$$H = h(u)\big(\cos\alpha(X^2 - Y^2) + 2\sin\alpha\, XY\big) \tag{4.6}$$

where $h(u)$ is an arbitrary function and α is the (constant) polarization of the wave and $\Psi_4 = h(u)e^{i\alpha}$.

It is sometimes convenient to introduce the concept of a polarization vector of the gravitational plane wave. Describing the space-time using the above notation, this may be considered to be a vector in the space-like wave surfaces that is inclined at the angle α to the X-coordinate direction.

For a gravitational wave with constant linear polarization it is possible to rotate the coordinates such that $h_{12} = 0$, or $\alpha = 0$. In this case the polarization vector is aligned with the X-coordinate direction, and the line element is then

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + h_{11}(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.7}$$

and $\Psi_4 = h_{11}(u)$. For a general plane gravitational wave, however, the function h_{12} will not be proportional to h_{11}, and the polarization will then be variable.

On the other hand, it can be seen that (4.4) describes a pure electromagnetic wave with zero Weyl tensor if $h_{22} = h_{11}$ and $h_{12} = 0$. It then takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + h_{11}(u)(X^2 + Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.8}$$

where $\Phi_{22} = h_{11}(u)$.

In order to consider the collision and interaction of plane waves of the above type, it will be found convenient to transform the general line element (4.4) to a form that was first considered by Rosen (1937). This uses two null coordinates u and v, and can be obtained from (4.4) using the transformation

$$\begin{aligned} X &= ax + by \\ Y &= ex + cy \\ r &= v + \tfrac{1}{2}(aa' + ee')x^2 + \tfrac{1}{2}(ba' + ab' + ec' + ce')xy + \tfrac{1}{2}(bb' + cc')y^2 \end{aligned} \tag{4.9}$$

where the coefficients a, b, c and e are all functions of u only, which are constrained by the equations

$$\begin{aligned} &a'' + h_{11}a + h_{12}e = 0 \\ &b'' + h_{11}b + h_{12}c = 0 \\ &c'' + h_{12}b + h_{22}c = 0 \\ &e'' + h_{12}a + h_{22}e = 0 \\ &ba' - ab' - ec' + ce' = 0. \end{aligned} \tag{4.10}$$

This transformation produces the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (a^2 + e^2)\mathrm{d}x^2 - 2(ab + ce)\mathrm{d}x\mathrm{d}y - (b^2 + c^2)\mathrm{d}y^2 \tag{4.11}$$

in which the coefficients are functions of u only. The curvature tensor components are now

$$\begin{aligned} \Phi_{22} &= -\frac{ca'' + ac'' - eb'' - be''}{2(ac - be)} \\ \Psi_4 &= -\frac{(ca'' - ac'' - eb'' + be'') - i(ba'' - ab'' + ec'' - ce'')}{2(ac - be)}. \end{aligned} \tag{4.12}$$

If the gravitational wave component has constant linear polarization, h_{12} in (4.4) may be put equal to zero, and b and e may also be taken as zero, producing a significant simplification.

It may be observed that there is a redundant function in the line element (4.11). It is therefore appropriate to rewrite it in an alternative form. For later convenience we choose the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - e^{-U}(e^V \cosh W\mathrm{d}x^2 - 2\sinh W\mathrm{d}x\mathrm{d}y + e^{-V}\cosh W\mathrm{d}y^2) \tag{4.13}$$

where U, V and W are functions of u only. In this form

$$\begin{aligned} \Phi_{22} &= \tfrac{1}{4}(2U_{uu} - U_u^2 - W_u^2 - V_u^2\cosh^2 W) \\ \Psi_4 &= -\tfrac{1}{2}(V_{uu}\cosh W - V_u U_u \cosh W + 2V_u W_u \sinh W) \\ &\qquad - \tfrac{1}{2}i(W_{uu} - W_u U_u - V_u^2 \cosh W \sinh W). \end{aligned} \tag{4.14}$$

In the case when the gravitational wave component has linear polarization, it is always possible to put $W = 0$.

That the above space-time describes plane waves has been demonstrated by Bondi, Pirani and Robinson (1959). They have shown that this class of solutions has plane symmetry and contains waves that propagate along the null hypersurfaces given by $u =$ constant. They have demonstrated that relative accelerations are produced in test particles when such a wave passes through them. They have also shown that the metric given by (4.13) admits a 5-parameter group of motions that are generated by the Killing vectors

$$\begin{aligned} \xi_1 &= \partial_x \\ \xi_2 &= \partial_y \\ \xi_3 &= \partial_v \\ \xi_4 &= x\partial_v + P_-(u)\partial_x + N(u)\partial_y \\ \xi_5 &= y\partial_v + P_+(u)\partial_y + N(u)\partial_x \end{aligned} \tag{4.15}$$

where

$$P_\pm(u) = \int e^{U\pm V} \cosh W \mathrm{d}u, \qquad N(u) = \int e^U \sinh W \mathrm{d}u. \tag{4.16}$$

All the corresponding operators commute except for $[\xi_1, \xi_4] = \xi_3$ and $[\xi_2, \xi_5] = \xi_3$. These commutators indicate that the structure constants for plane gravitational waves are the same as those for plane electromagnetic waves in a flat space-time. This analogy with plane electromagnetic waves further justifies their interpretation as plane waves.

4.3 Particular cases

In the above discussion, some attention has already been paid to the cases of pure electromagnetic waves, and pure gravitational waves with linear polarization. These have been described by the line elements (4.8) and (4.7), but the profile of the waves, which is determined by the arbitrary function $h_{11}(u)$, has been left totally general. In this section, some waves with specific profiles will be considered in a little more detail. The purpose here is to describe some of the properties of the waves that will be considered in the following chapters.

Consider first a shock electromagnetic wave with a step profile

$$\Phi_{22} = a^2 \Theta(u). \tag{4.17}$$

For this the line element (4.8) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2 + Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.18}$$

In transforming this to Rosen form, the region in front of the wave is flat and, for $u < 0$, has the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{4.19}$$

In the region $u \geq 0$, which contains the electromagnetic wave, the line element is

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{4.20}$$

Notice that, in this form, the metric is differentiable across the wave front $u = 0$.

Consider similarly a gravitational step wave with

$$\Psi_4 = a^2\Theta(u). \tag{4.21}$$

The line element (4.7) becomes

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a^2\Theta(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2. \tag{4.22}$$

The region in front of the wave is flat having the line element (4.19), but for $u \geq 0$ we obtain

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 au\,\mathrm{d}x^2 - \cosh^2 au\,\mathrm{d}y^2. \tag{4.23}$$

Finally, we may point out that the case of an impulsive gravitational wave has already been described in Section 3.1. If $\Psi_4 = a\delta(u)$, the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + a\delta(u)(X^2 - Y^2)\mathrm{d}v^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{4.24}$$

can be transformed to

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1 - au)^2\mathrm{d}x^2 - (1 + au)^2\mathrm{d}y^2 \tag{4.25}$$

in the region following the wave front. For $u < 0$, the line element is (4.19) as before.

This example of an impulsive gravitational wave is particularly important since an impulsive wave can be considered in some sense as an idealization of any wave of finite duration. It may be noticed that the metric (4.25) is actually flat. The curvature components occur only on the wave front $u = 0$. Thus, this line element can also be used, with a possible linear transformation of the coordinate u, to describe the region following any sandwich gravitational wave with constant linear polarization.

4.4 Global properties

It can be seen that the general line element (4.4), including the particular cases (4.18), (4.22) and (4.24), describes a space-time containing plane waves. The same form of the metric can also describe a sandwich wave and the regions both in front of and behind it. The wave, however, is of infinite extent in all directions in its plane. The energy of the wave is therefore, in a global sense, infinite.

One further problem of this Brinkmann form of the metric is that discontinuities arise in the metric components. This can be seen explicitly in (4.18), (4.22) and (4.24). However, this difficulty can easily be removed since it is possible to transform these line elements to the Rosen form (4.11) or (4.13), which is always continuous. The discontinuities in the Brinkmann form of the metric arise from the particular choice of coordinates.

On the other hand, it can also be seen that the line elements in Rosen form (4.20), (4.23) and (4.25) are singular. They all have coordinate singularities on some hypersurface behind the wave. As will be described in the following chapter, this can be interpreted in terms of the focusing of the null congruences opposing the waves that are associated with the coordinate v. For single plane waves, these are only coordinate singularities. In fact, coordinate singularities of various kinds inevitably arise whenever the metric is taken in Rosen form (4.11) or (4.13).

The focusing property of plane waves has been further analysed by Bondi and Pirani (1989) and described in terms of caustics. They have considered the effect of a sandwich plane gravitational wave on a set of test particles that are strung out in a fixed direction, which depends on the polarization of the wave, and are initially at rest in a flat space-time. They have proved, at least for waves that have constant linear polarization, that all the particles will collide after a finite time independent of how far apart they were initially. The occurrence of such caustics has been described in detail. They are associated with the coordinate singularities that occur in the Rosen form of the metric. For pure electromagnetic waves, a whole three-dimensional set of basic world lines may pass through a single event, as also pointed out by Cantoni (1971). This is one aspect of the focusing power of gravitational and electromagnetic waves. Further aspects will be discussed in the following chapters.

Bondi and Pirani (1989) have also considered further geometrical properties of plane waves as seen by an observer who passes through the wave. Since the world lines of initially stationary test particles may meet a finite time after passing through a wave, however far apart they were initially, the observable pasts of both particles will eventually coincide. It follows that, after passing a wave front, an observer will, within a finite time, have seen the whole of an infinite spatial volume in a strip of the half hyperplane in front of the wave.

Further properties of plane waves have been described by Penrose (1965*a*) from a different point of view. He has considered the structure of the future light cone of a point Q in front of the wave. As this cone expands, part of it crosses the wave and is distorted by it. Eventually it will encounter the coordinate singularity behind the wave. The light emitted from Q that passes through the wave will be focused by it. This process will be described in the next chapter. For a gravitational wave the light will be focused to a line, but for a pure electromagnetic wave the light will be focused to a point. In either case the light will appear on the past light cone of this line or point, which may be denoted by R. It can thus be seen that the future light cone of a point Q in front of the wave is identical to the past light cone of the "point" R behind the wave. This is illustrated in Figure 4.1.

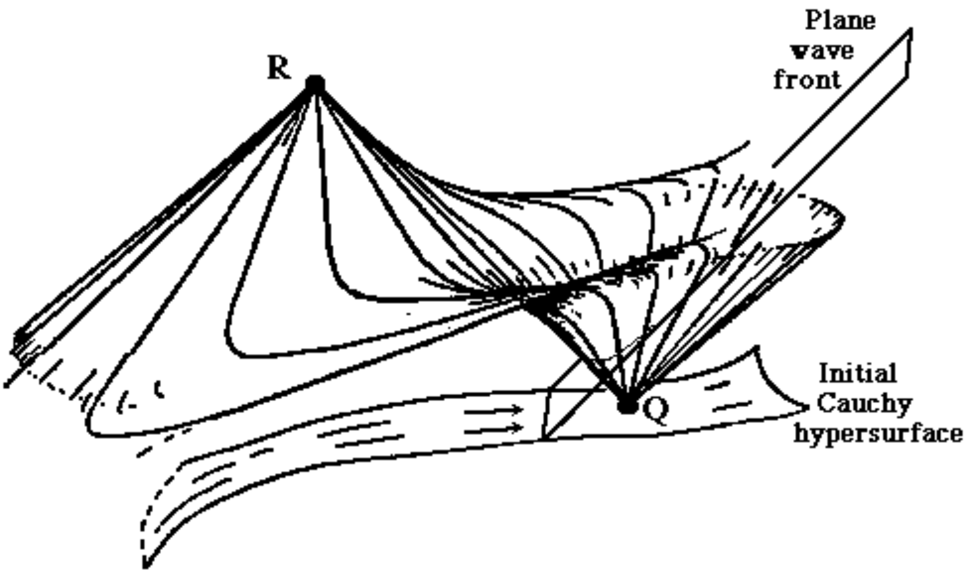

Figure 4.1 (Penrose 1965a) The future light cone of the point Q is distorted as it passes through a plane wave and is again focused to another vertex R which may be a point or a line (one spatial dimension has been suppressed).

Another remarkable property of plane waves can be deduced immediately from Figure 4.1, as has been argued in detail by Penrose (1965a). It implies that a plane wave space-time contains no global Cauchy hypersurface. In other words, it implies that it is not possible to set up initial values for a plane wave on any global space-like hypersurface. Such a hypersurface must lie entirely in the past of any future null cone from any point on the surface. However, in this case, the future null cone of Q folds down to focus again at R. The Cauchy hypersurface passing through Q must therefore lie entirely below the past null cone of R. It can not, therefore, extend to spatial infinity behind the wave. Thus no space-like hypersurface exists that is adequate for the global specification of Cauchy data.

It also follows from the above discussion that it is not possible to embed a plane wave globally in any hyperbolic normal pseudo-Euclidean space.

It may be noted that a further mathematical description of the geometrical properties of plane waves, particularly related to colliding wave problems, has also been given by Yurtsever (1988*b*). These properties will be also described in more detail in some of the following chapters.